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Optimal Randomized Complete Visibility on a Grid for Asynchronous Robots with Lights

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Abstract

We consider the distributed setting of N autonomous mobile robots that operate in *Look-Compute-Move* (LCM) cycles and communicate with other robots using colored lights (the *robots with lights* model). This model assumes *obstructed visibility* where a robot cannot see another robot if a third robot is positioned between them on the straight line connecting them. In this paper, we consider robot movements to be on a grid (integer plane) of unbounded size. In any given step a robot positioned at a grid point can move only to an adjacent grid point to its north, south, east or west. The grid setting naturally discretizes the 2-dimensional plane and finds applications in many real-life robotic systems. The Complete Visibility problem is to reposition N autonomous robots (starting at arbitrary, but distinct, initial positions) so that, on termination, each robot is visible to all others. The objective in this problem is to simultaneously minimize (or provide trade-off between) two fundamental performance metrics: (i) time to solve Complete Visibility and (ii) area occupied by the solution. We also consider the number of distinct colors used by each robot light. We provide the first $\mathcal{O}(\max\{D, N\})$-time algorithm for Complete Visibility in the asynchronous setting, where D is the diameter of the initial configuration. The area occupied by the final configuration is $\mathcal{O}(N^2)$; both the time and area are optimal. The time is randomized if no symmetry breaking mechanism is available for the robots. The number of colors used in our algorithm depends on whether leader election is required or not: (i) 17 colors if leader election is not required and (ii) 32 colors if leader election is required.

1 Introduction

The Classical Model. The classical model of distributed computing by mobile robots models each robot as a point in the plane [7]. The local coordinate system of a robot may not be consistent with that of other robots. A robot can determine the positions of other (visible) robots in its own coordinate system. The robots are *autonomous* (no external control), *anonymous* (no unique identifiers), *indistinguishable* (no external identifiers), and *disoriented* (no agreement on local coordinate systems). They execute the same algorithm. In the classical model, robots have *unobstructed* visibility, that is, robots are transparent such that all robots can see each other regardless of their position. Each robot proceeds in *Look-Compute-Move* (LCM) cycles: When a robot becomes active, it first gets a snapshot of its surroundings (*Look*), then computes a destination based on the snapshot (*Compute*), and finally moves to the destination (*Move*). Moreover, the robots are *oblivious* and *silent* – each robot has no memory of its past LCM cycles and they do not communicate directly [7].

Robots with Lights. The *robots with lights* model [3, 7, 13] incorporates direct communication – each robot has a visible light that can assume colors from a constant-sized set; robots explicitly communicate with each other using these colors. The colors are *persistent*; i.e., the color is not erased at the end of a cycle. Except for the lights, the robots are oblivious as in the classical model. Also, robots have *obstructed visibility* – robot b placed between robots a, c on the line connecting them blocks a (resp., c) from seeing c (resp., a).

Complete Visibility. The fundamental Complete Visibility problem is defined as follows: Given an arbitrary initial configuration of N autonomous mobile robots located at distinct points, they reach a configuration in which each robot is at a distinct point from which it can see all others (i.e., no three robots are collinear). Initially, some robots may be obstructed from the view of others (obstructed visibility). The robots do not know the total number of robots, N, and they do not have any agreement on a global coordinate system. Solving Complete Visibility enables solutions to many other robot coordination problems under obstructed visibility, e.g., gathering, pattern formation, and leader election. Indeed, after solving Complete Visibility, each robot is able to see all the robots, a scenario similar to having unobstructed visibility. Complete Visibility has been receiving much attention recently. The objective in this paper is to simultaneously minimize two fundamental performance metrics: (i) time to solve Complete Visibility and (ii) area occupied by the robots in the final configuration. Typically, the number of colors used is also an important consideration. The area occupied has been less studied. A series of papers resulted in a deterministic $\mathcal{O}(1)$-time, 47-color algorithm in the 2-dimensional real plane in the asynchronous setting [17, 19–21]. Hector and Vaidyanathan [10] established the area (or spatial complexity) of these algorithms to be optimal.

Grid. In this paper, we consider solving Complete Visibility in the robots with lights model on a *grid* of unbounded size embedded in the 2-dimensional plane that restricts a single movement of a robot to one of the four neighboring grid points. Further, the set of grid points is known to all robots. Informally, one could think of the robots being given the grid lines on the plane. From this information, each robot can readily determine unit distance and the collective directions of the two grid axes. Each robot has a local sense of the individual axes' directions and their orientations, but this is not globally common to all robots.

The grid setting naturally discretizes the 2-dimensional plane and finds applications in real-life robot navigation systems, such as industrial Automated Guided Vehicles [12] and Coverage Path Planning [16]. The movement of robots along grid lines from one grid point to its adjacent point can be easier to implement for robots with weak capabilities as they may not be able to execute accurate movements in arbitrary directions or by arbitrarily small amounts [1]. Even with advanced robot navigation capabilities, the resolution of its plane of movement is finite, and consequently, a countable plane (rather than a real plane) better models the robots' movements; indeed, many algorithms for the real plane assume infinite resolution by, for example, relying on sufficient room to accommodate any number of robots between two given robots (regardless of how close they are to each other). Further, the asynchronous setting, which we assume, also admits differences between individual robots, whether that is in the form of minor variations (in a homogeneous swarm) or

Algorithm	**Time (in epochs)**	**Area**	**Number of colors**	**Remarks**
Adhikary *et al.* [1]	$\Omega(\max\{DN, N^2\})$ (**Th. 2.4**)	$\Omega(N^2)$ (**Th. 2.5**)	11	leader election not needed, deterministic
Lower bound (this paper)	$\Omega(N)$ (**Th. 2.6**)	$\Omega(N^2)$ (**Th. 2.5**)	any	any
Upper bound (this paper)	$\mathcal{O}(\max\{D, N\})$	$\mathcal{O}(N^2)$ (**Th. 1.1**)	17	if leader election not needed, deterministic
Upper bound (this paper)	$\mathcal{O}(\max\{D, N\})$	$\mathcal{O}(N^2)$ (**Th. 1.1**)	32	if leader election needed, randomized

Table 1: Complete Visibility of $N \geq 1$ robots on a grid of unbounded size in the asynchronous setting; D is the diameter of the initial configuration. The randomized algorithm terminates in the stated time with probability at least $1 - 2^{-\max\{D,N\}}$.

large-scale differences (in a heterogeneous swarm).

Contributions. We consider the same robot model as Di Luna *et al.* [5], namely, robots are oblivious except for a persistent light that can pick a color at a time from a constant-sized set. Robots can have their visibility obstructed by other robots in the line of sight, robots do not know the total number of robots, N, and the robots are disoriented (no agreement on global coordinate system). Moreover, the robot setting is *asynchronous* – they have no common notion of time and robots perform their LCM cycles at arbitrary times (this idea is further elaborated upon in Section 2). The robots operate on an infinite grid, where each grid point connects to four other (neighboring) grid points, each at unit distance away (to its "North," "South," "East," and "West" in its local coordinate system). A single robot movement is restricted to be to one of its neighbors at unit distance. Two robots should not occupy the same grid point simultaneously (this would constitute a *collision*). Runtime is measured in *epochs* – the interval where every robot completes at least one LCM cycle. We prove the following main result which, to our knowledge, is the first algorithm with optimal runtime and area for Complete Visibility on a grid (of unbounded size).

Theorem 1.1 (main result) *For any initial configuration of $N \geq 1$ robots with lights on distinct points on a grid of unbounded size,* Complete Visibility *can be solved optimally in $\mathcal{O}(\max\{D, N\})$ time (where D is the diameter of the initial configuration), and with a final optimal area of $\mathcal{O}(N^2)$. Further, the algorithm runs in the asynchronous setting without collisions using:*

- *17 colors through a deterministic algorithm if leader election is not required; and*
- *32 colors through a randomized algorithm which terminates in the given time bounds with probability at least $1 - \frac{1}{2^{\max(D,N)}}$ if leader election is required.*

The following two results further emphasize the significance of Theorem 1.1.

a. A runtime of $\Omega(\max\{DN, N^2\})$ for the deterministic algorithm of Adhikary *et al.* [1] for Complete Visibility on a grid of unbounded size (**Theorem 2.4**).

b. A time lower bound of $\Omega(N)$ and an area lower bound of $\Omega(N^2)$ for Complete Visibility on a grid of unbounded size (**Theorems 2.5 & 2.6**), which hold even if an unlimited number of colors is available and leader election is not required (i.e., a unique leader is already provided or identified). In other words, this time lower bounds holds for both deterministic and randomized algorithms.

Note that the randomized algorithm in the context of this paper is the algorithm that uses randomized leader election procedure. Therefore, except leader election, the rest is deterministic even for our randomized algorithm.

All the above results are summarized and compared with the closest previous work in Table 1. In summary, Theorem 1.1 improves runtime significantly compared to Adhikary *et al.* [1]. In fact, our algorithm is asymptotically time-optimal for $D = \mathcal{O}(N)$ (Theorem 2.6). Moreover, Theorem 1.1

bounds for the first time the area of the final configuration of the solution. In fact the area occupied by the robots in our algorithm is optimal for any D (Theorem 2.5). Additionally, our time lower bound shows that irrespective of (i) whether leader election is needed and (ii) whether there is a limit on the number of colors, the time bound cannot be better than $\mathcal{O}(N)$ and the area bound cannot be better than $\mathcal{O}(N^2)$. Therefore, our result provides trade-off on the number of colors and nature of the algorithm based on whether leader election is required.

Note: This is an extended, full version of a preliminary version appeared in APDCM'20 [18]. This version includes proofs as well as many details that were omitted in the preliminary version. Additionally, the number of colors in our algorithm with leader election is improved from 50 (reported in the preliminary version) to 32 in this version, through a detailed analysis on the number of colors.

Techniques. The area lower bound proof uses an argument based on the well-known pigeonhole principle. For the time upper bound, our proposed algorithm has three stages, Stages 1–3, that execute one after another. Stage 1 is needed only when a leader needs to be identified (i.e., a leader is not provided already). Stage 1 (if executed) elects two robots as the first and second leaders satisfying some properties. Stage 2 moves all the robots to position them on consecutive grid points on an axis-aligned line connecting the leaders. Stage 3 moves all robots except the first leader to position themselves on grid points in a manner that solves Complete Visibility. Keys to Stage 1 are *corner moving* and *leader election* procedures that allow electing two robots as leaders with certain desired properties. Key to Stage 2 is a *line formation* procedure that positions all N robots on consecutive positions on an axis-aligned line connecting the leaders. Key to Stage 3 is a *placement* procedure that positions the robots in grid points guaranteeing Complete Visibility.

We will show that each stage finishes in $\mathcal{O}(\max\{D, N\})$ time, giving overall $\mathcal{O}(\max\{D, N\})$ runtime. Stage 1 for leader election finishes in this time with probability at least $1 - \frac{1}{2^{\max\{D,N\}}}$ (and makes the overall algorithm randomized). Stages 2 and 3 are deterministic. If leader election is unnecessary, Stage 1 is not needed, and hence the overall algorithm is deterministic. The total number of colors used throughout the stages with leader election is 32 and without leader election is 17.

Remark on Leader Election Requirement. Since robots are indistinguishable, deterministic leader election is not possible without some other mechanism to break symmetry. For instance, suppose robots have a compass, i.e., robots agree on North (top), South (bottom), East (right), and West (left) directions. This immediately provides the leader (for example, pick the southernmost robot; break a tie by picking the westernmost among all southernmost ones).

Closely Related Work. Adhikary *et al.* [1] gave the only Complete Visibility algorithm for robots with lights operating on a grid of unbounded size. All other Complete Visibility algorithms are for the real plane (as noted earlier). The algorithm of Adhikary *et al.* is deterministic and uses 11 colors in the asynchronous setting. While their work proves algorithm correctness, it does not provide a runtime and area analysis, except for a proof of finite time termination. Our analysis of their algorithm gives $\Omega(\max\{DN, N^2\})$ runtime, where D is the diameter of the initial configuration. We do not have analysis on the area occupied by the final configuration of the robots in Adhikary *et al.* but what is known is that the area is at least $\Omega(N^2)$. In contrast, our algorithm runs in $\mathcal{O}(\max\{D, N\})$ time with very high probability and the final configuration has an area of $\mathcal{O}(N^2)$. Both the time and area are asymptotically optimal. Some of the results (particularly line formation in Stage 2) are recently used by Hector *et al.* [11] for the convex hull formation problem on the grid of unbounded size and provided (asymptotically optimal) time and perimeter bounds.

Many papers focus on solving Complete Visibility on the 2-dimensional real plane. Di Luna *et al.* [5] gave the first algorithm for robots with lights. Subsequent papers [4, 15] minimized the number of colors. Vaidyanathan *et al.* [21] presented an algorithm with runtime $\mathcal{O}(\log N)$ using 12 colors in the fully synchronous setting. After that, Sharma *et al.* [19] presented an algorithm with runtime $\mathcal{O}(1)$ using 12 colors in the semi-synchronous setting. Finally, Sharma *et al.* [17, 20] then presented an algorithm with runtime $\mathcal{O}(1)$ using 47 colors in the asynchronous setting.

Furthermore, Complete Visibility on a grid has been considered in the literature as the No-Three-In-Line problem [6, 8, 9], where the goal is to select grid points from an $N \times N$ grid so that no three selected points are collinear. The maximum number of such grid points is $2N$ for $N \leq 46$, and for $N >> 46$, the best solution known is $(\frac{3}{2} - \epsilon)N$, for any $\epsilon > 0$ [9]. However, these works do not deal with relocating robots to place themselves on the grid points to satisfy a No-Three-In-Line configuration. This becomes further challenging when (i) robots do not know N, (ii) robots and grid both are anonymous, and (iii) robots are disoriented.

Roadmap. The rest of the paper is organized as follows. Section 2 provides details on the model, the runtime of the previous algorithm, and the runtime and area lower bounds of $\Omega(N)$ and $\Omega(N^2)$, respectively, for any Complete Visibility algorithm on a grid. We present our $\mathcal{O}(\max\{D, N\})$-time, $\mathcal{O}(N^2)$-area algorithm and its analysis in Section 3 proving Theorem 1.1. The algorithm is deterministic or randomized depending on whether leader election is required or not. Section 4 concludes the paper with a short discussion.

2 Model and Preliminaries

Robots. We consider a system of N robots (agents) $\mathcal{Q} = \{r_0, r_1, \cdots, r_{N-1}\}$. Each robot is a (dimensionless) point that can move in a 2-dimensional grid $\mathbb{Z}^2$ of unbounded size embedded in the plane (here $\mathbb{Z}$ is the set of integers). All the robots are initially positioned on grid points. The robots do not have access to any global coordinate system (except as noted in the description of the "Grid" in Section 1) and they do not know the value of N. Robots agree on the location of grid points, but each robot's orientation of up/down, left/right in the grid is local and is independent of the orientation of other robots. A robot r_i can see, and be visible to, another robot r_j iff there is no third robot r_k in the line segment joining r_i and r_j (note that this line segment does not have to be axis-aligned). Each robot has a light that can assume one color at a time from a constant-sized set. We will often use r_i to denote a robot as well as its position.

Look-Compute-Move. At any time, each robot r_i is either active or inactive. When a robot r_i becomes active, it performs the "Look-Compute-Move" cycle described as follows. (i) *Look:* For each robot r_j that is visible to it, r_i can observe the position of r_j on the grid and the color of the light of r_j. Robot r_i can also observe its own color and position. Each robot observes position in its own frame of reference. That is, two different robots observing the position of the same point may produce different coordinates. A robot observes the positions of points accurately within its own reference frame. (ii) *Compute:* Robot r_i may perform an arbitrary computation using only the colors and positions observed during the "look" portion of that cycle. This includes determination of a (possibly) new position (at a neighboring grid point) and light color for r_i. (iii) *Move:* Robot r_i changes its light to the new color (if computed) and moves to its new position.

Robot Movement. A robot's movement is restricted to grid lines. In other words, from a robot's current grid point, it can move only to one of the four neighboring grid points in one move. For simplicity in analysis, we assume that the moves are *instantaneous*, i.e., the robots are always seen at grid points (not on grid edges). Numerous grid and graph algorithms make this assumption (for instance, [1, 7]).

Robot Activation and Synchronization. In the fully synchronous setting ($\mathcal{FSYNC}$), every robot is active in every synchronized LCM cycle. In the semi-synchronous setting ($\mathcal{SSYNC}$), at least one robot is active in each synchronized LCM cycle, and over an infinite number of LCM cycles, every robot is active infinitely often. In the asynchronous setting ($\mathcal{ASYNC}$), robots have no common notion of time. No limit exists on the number and frequency of LCM cycles in which a robot can be active except that every robot is active infinitely often. Complying with the $\mathcal{ASYNC}$ setting, we assume that a robot "wakes up" and performs its Look phase instantaneously. An arbitrary amount of time may elapse between Look/Compute and Compute/Move phases.

Runtime and Area. For the $\mathcal{FSYNC}$ setting, time is measured in rounds. Since a robot in the $\mathcal{SSYNC}$ and $\mathcal{ASYNC}$ settings could stay inactive for an indeterminate time, we use the idea of an epoch to measure time. An *epoch* is the smallest time interval within which each robot is active at least once [2]. Let t_0 denote the start time. Epoch i ($i \geq 1$) is the time interval from t_{i-1} to t_i where

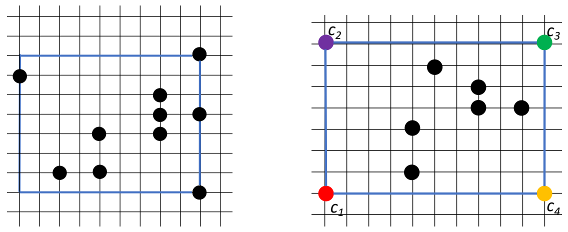


Figure 1: Examples: (a) Smallest Enclosing Rectangle (SER), (b) Four-Corners Configuration (FCC)

t_i is the first time instant after t_{i-1} when each robot has finished at least one complete LCM cycle. Therefore, for the $\mathcal{FSYNC}$ setting, a round is an epoch. We will use "time" generically to mean rounds for the $\mathcal{FSYNC}$ and epochs for the $\mathcal{SSYNC}$ and $\mathcal{ASYNC}$ settings.

For area, we measure the size of the axis-aligned rectangle that encloses all the robots in their final configuration solving Complete Visibility.

Smallest Enclosing Rectangle (SER) and Four-Corners Configuration (FCC). A *smallest enclosing rectangle* (SER) of a configuration is an axis-aligned rectangle with minimum area such that all robots are on its perimeter or in its interior. A *four-corners configuration* (FCC) is a configuration in which one robot is at each of the four corners and the remaining $N-4$ robots are in the interior. For $N = 10$, Fig. 1 shows an SER (left) and an FCC (right).

Line and Line Segment. For any pair of points a, b, we denote the line segment connecting them by $\overline{ab}$ and its length by $\mathsf{length}(\overline{ab})$. We denote a line with one endpoint a and passing through b by $\overrightarrow{ab}$. When we refer to grid points on $\overrightarrow{ab}$, we always refer to points on $\overrightarrow{ab}$ starting from a towards b (and possibly beyond b).

Configuration. A *configuration* $\mathbf{C}_t$ specifies the robots' positions and colors at time $t \geq 0$. Formally, $\mathbf{C}_t = \left\{\left(pos_i^t, col_i^t\right) : 0 \leq i < N\right\}$ for robot positions pos_i^t and colors col_i^t at time t. A configuration, $\mathbf{C}_t(r_i)$, for a robot $r_i \in \mathcal{Q}$ at time t is a restriction of $\mathbf{C}_t$ to those robots that are visible to r_i at t. Clearly, $\mathbf{C}_t(r_i) \subseteq \mathbf{C}_t$. We will sometimes write $\mathbf{C}, \mathbf{C}(r_i)$ to denote $\mathbf{C}_t, \mathbf{C}_t(r_i)$.

Basic results. The following results will be used in the algorithm in Section 3.

Lemma 2.1 *Let L_i be an axis-aligned grid line passing through robot $r_i \in \mathcal{Q}$ such that: (a) there is at least one robot on each side of L_i, and (b) r_i can see a robot colored* `red` *on only one side of L_i (with no visible* `red`*-robots on the other side). Let LP_i be a line perpendicular to L_i passing through r_i. Let p_{up} and p_{down} be the grid points on LP_i adjacent to r_i's position where p_{up} is on the side with* `red` *robots. Robot r_i can correctly move to p_{up} or p_{down} as needed.*

Proof. Since robots on one side of L_i are colored differently than the other side, r_i can differentiate the neighboring points p_{up} or p_{down} on LP_i as needed. □

Lemma 2.2 *Let L_i be an axis-aligned grid line passing through robot $r_i \in \mathcal{Q}$. Let LP_i be a line perpendicular to L_i passing through r_i. Let LP_f and LP_j be lines parallel to LP_i and on opposite sides of LP_i. Let L_α be a line parallel to L_i such that a robot each from LP_f and LP_j is on it with color different from other robots on those lines. Let there be no robots between LP_f and LP_i and between LP_i and LP_j on the side of L_i that includes L_α, and let there be no robot on LP_i or LP_j beyond L_α. Let p be the grid point on LP_i adjacent to r_i's position and on the side of L_i opposite that of L_α. Robot r_i can correctly move to p.*

Proof. Since a robot r_i on LP_i always sees robots on LP_f and LP_j colored different from other robots, by Lemma 2.1, r_i can correctly move to point p. □

Lemma 2.3 *Let $\overline{AB}$ be an axis-aligned grid line segment joining two grid points A, B. Suppose $\mathsf{length}(\overline{AB}) = x$ units. For any $1 \leq y \leq x+1$, y robots placed arbitrarily on the points of $\overline{AB}$ can be relocated to be positioned consecutively on $\overline{AB}$ starting from A (or B) in $2x+2$ epochs.*

Proof. If there are $y = x + 1$ robots on $\overline{AB}$, no relocation is necessary. If $y = 1$, then it needs x epochs. Suppose the only robot is at B and has to move to A. If there are $2 \leq y \leq x$ robots, then a robot may not be able to make its first move for $y - 1$ epochs (suppose y robots are on consecutive positions). After that, it can move in each epoch as the robots are moving in the same direction and when the next point is empty and a robot moves into it, then the new next point will be empty in the next epoch as well. Therefore in total, $y \leq 2x + 2$ epochs. □

Time Complexity of the Previous Algorithm. Adhikary *et al.* [1] constructed the only previous algorithm for Complete Visibility on a grid. They provided no runtime.

Theorem 2.4 (runtime of Adhikary *et al.* **[1])** *The algorithm of Adhikary et al. [1] for* Complete Visibility *on a grid of unbounded size has runtime* $\Omega(\max\{DN, N^2\})$.

Proof. Their algorithm has three phases that execute sequentially.

- **Phase 1 (interior depletion)** moves all the robots in the interior of the initial configuration to form an SER configuration so that all N robots are positioned on the boundary of the SER. The robots already on the boundary of the SER do not move and the robots in the interior of the SER sequentially move to position themselves on the boundary of the SER. If all positions on the SER boundary are occupied, then the robots on the boundary of the SER move to the exterior to create empty positions. After all the robots position themselves on the boundary of the SER, Phase 1 finishes.

- **Phase 2 (corner creation)** If the four robots on the SER are not already on the four corner points of the SER, then robots on the boundary of the SER move to place robots on the four corners of the SER. Phase 2 then finishes.

- **Phase 3 (symmetric movements)** moves the robots on the SER, two at a time (from each side of the SER), in the exterior of the SER to form a Complete Visibility configuration. After two robots (from a side of SER) are placed on their positions, the next two from that side will move, and so on. The moves of two robots from four sides of SER run independently in parallel. The four robots on the corners of the SER will not move, and they act as references to position the $N - 4$ robots appropriately to achieve a Complete Visibility configuration. If two robots moved simultaneously before are placed at distance d from the Phase 2 SER (in the exterior of the Phase 2 SER), the two robots moved simultaneously after are placed at distance $d' > d$.

We now discuss why the runtime is $\Omega(\max\{DN, N^2\})$. In Phases 1 and 3, there are configurations such that these phases need time $\Omega(DN)$ and $\Omega(N^2)$, respectively. For time $\Omega(DN)$ for Phase 1, consider the initial configuration of $\lfloor N/2 \rfloor$ robots in the interior on a line (that is not a vertical or horizontal axis-aligned line of the grid) and the remaining $\lceil N/2 \rceil$ robots on the SER. This gives $\lfloor N/4 \rfloor$ rectangle layers in the interior of the SER with two robots serving as the corners of each rectangle layer. Suppose the distance between robots on the SER boundary to any robot in the interior of it is $D = \Omega(N)$. In this configuration, the robots in the interior of the SER need to move to the SER boundary. Since the robots on a side move sequentially, only two robots on a rectangle layer in the interior of the SER can move to the SER at any round. Since there are $\lfloor N/2 \rfloor$ robots in the interior and the distance from each of them to G' is $\Omega(D)$, the total time is $\Omega(DN)$.

For time $\Omega(N^2)$ for Phase 3, consider an SER with $N/4$ robots on each side. Only two robots from each side move simultaneously at any time. After they are settled, then two other robots move. This way this process repeats $N/8$ times. Two robots settle at distance 1 from the Phase 2 SER, two robots settle at distance 2 from the Phase 2 SER, and so on. This way, the last two robots have to move $N/8$ distance. Therefore, the total time is $\geq (1+2+\ldots+N/8) \geq N/8(N/8+1)\cdot 1/2 = \Omega(N^2)$. □

Time and Area Lower Bounds. In the grid plane, we consider the area of a configuration to be the area of the SER of that configuration. The following results establish lower bounds the time and area of the final configuration of any Complete Visibility algorithm for the grid.

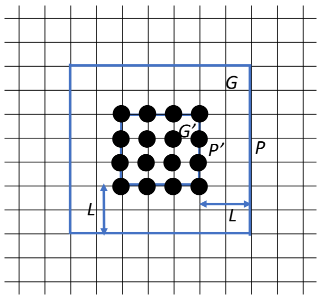


Figure 2: Lower bound example

Theorem 2.5 Every Complete Visibility algorithm for N robots running on a grid has a final configuration whose SER has each side of size $\Omega(N)$ and whose area is $\Omega(N^2)$.

Proof. In any solution to Complete Visibility no three robots can be on a straight line. This implies that each "horizontal" or "vertical" grid line can hold at most two robots. That is, the number of horizontal (or vertical) lines included in the SER of the final configuration is at least $\lceil \frac{N}{2} \rceil = \Omega(N)$. Therefore the area is at least $\lceil \frac{N}{2} \rceil^2 = \Omega(N^2)$. □

We now use Theorem 2.5 to establish that there exists an initial configuration from which at least one robot must move $\Omega(N)$ times; recall that a Complete Visibility algorithm must admit any initial configuration.

Theorem 2.6 Every Complete Visibility algorithm for N robots running on a grid that admits an arbitrary initial configuration has a worst-case time complexity of $\Omega(N)$.

Proof. Consider an initial configuration in which the N robots are placed compactly in a $\left\lceil \sqrt{N} \right\rceil \times \left\lceil \sqrt{N} \right\rceil = A_i \times A_i$ SER S_i (say). (See Fig. 2.) By Theorem 2.5, the Complete Visibility algorithm must move the robots into an SER S_f of size $A_f \times B_f = \Omega(N) \times \Omega(N)$. In a final configuration whose SER is S_f, each side of S_f must have at least one robot positioned on it. Regardless of where S_i is positioned relative to S_f, it is easy to see that there is at least one side of S_f all of whose points are at least $\frac{1}{2}\min\{A_f, B_f\} - A_i = \Omega\left(N - \sqrt{N}\right) = \Omega(N)$ distance from every point in S_i. Therefore, at least one robot must move $\Omega(N)$ distance and the lower bound on the time of the algorithm is $\Omega(N)$. □

Observe that the results of Theorems 2.5 and 2.6 are independent of whether robots have lights or not and their synchronization mode. These theorems will be used to establish the proposed algorithm to be optimal in time and area.

3 $\mathcal{O}(\max\{D, N\})$-Time, $\mathcal{O}(N^2)$-Area Algorithm

In this section, we describe an $\mathcal{O}(\max\{D, N\})$-time, $\mathcal{O}(N^2)$-area algorithm for Complete Visibility on a grid of unbounded size in the $\mathcal{ASYNC}$ setting for robots with lights. For simplicity in the analysis, we first consider the cases of $D = \mathcal{O}(N)$. The algorithm and analysis extend immediately, replacing N by D for cases where $D = \Omega(N)$. The area bound remains independent of D. The algorithm consists of three stages, Stages 1–3. Fig. 3 provides an overview of these stages.

- **Stage 1 (leader election)** positions robots on the corners of an FCC and elects two robots A and B on a side of the FCC to be leaders. A is elected first and is called the "first leader". B is elected after A and is called the "second leader". The remaining two robots C, D on a side of the FCC are also ranked first and second. Stage 1 is not needed if leader election capability is already provided or available.

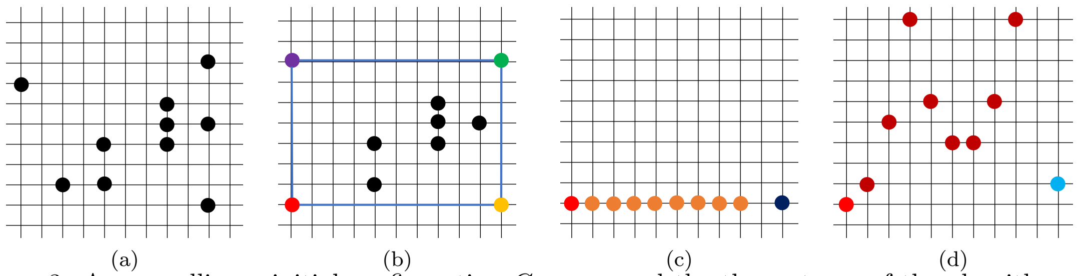


Figure 3: A non-collinear initial configuration $C_{non,init}$ and the three stages of the algorithm: (a) $C_{non,init}$, (b) after Stage 1, (c) after Stage 2, and (d) after Stage 3. The configuration after Stage 3 is a Complete Visibility configuration.

- **Stage 2 (line formation)** moves the $N-2$ non-leader robots to position themselves on consecutive grid points on line $\overrightarrow{AB}$ starting from A towards B, except one robot that will be positioned on $\overrightarrow{AB}$ at the grid point that is at distance $N'-1$ from A, where $N' \geq N$ is the first prime number greater than or equal to N.

- **Stage 3 (placement)** moves all robots (except A) from $\overrightarrow{AB}$ perpendicular to $\overrightarrow{AB}$ to position them modulo N' on grid points forming a Complete Visibility configuration, where Theorem 3.14 due to Roth [14] guarantees visibility.

At the initial configuration $\mathbf{C}_{init}$, all robots have color `start`. Each robot r_i works autonomously having only the information $\mathbf{C}(r_i)$. Though robots are oblivious, the colors and configurations of robots synchronize execution of the stages so that robots execute stages sequentially one after another.

We differentiate initial configurations $\mathbf{C}_{init}$ into two categories, collinear initial configurations $C_{col,init}$, and non-collinear initial configurations $C_{non,init}$. $\mathbf{C}_{init}$ is $C_{col,init}$ if all the robots are on a single grid line in $\mathbf{C}_{init}$, otherwise $\mathbf{C}_{init}$ is $C_{non,init}$. An important point to note in the definition of $C_{col,init}$ is that it does not include configurations where all robots exist on a single line that is not a grid line. For example, in Fig. 1(b), if all robots are on line $\overline{c_1c_4}$ or $\overline{c_3c_4}$, then it is $C_{non,init}$, but if all robots are on line $\overline{c_1c_3}$ then it is not $C_{non,init}$. In both $C_{col,init}$ and $C_{non,init}$, Stage 3 is common. For $C_{col,init}$, Stages 1 and 2 can be executed together. Therefore, Sections 3.1 and 3.2 describe Stages 1 and 2 for $C_{non,init}$, respectively, and Section 3.3 describes Stages 1 and 2 together for $C_{col,init}$. Finally, Section 3.4 describes Stage 3 with correctness and runtime.

3.1 Stage 1: Leader Election for Non-collinear Configurations

Stage 1 forms a four-corners configuration (FCC) among robots and designates two of the corner robots A and B on one side as leaders. We first discuss how to elect A and B for any $C_{non,init}$ when $N \geq 4$. We then discuss a special case of $C_{non,init}$ when $N = 3$. We have two sub-stages, Stage 1.1 and Stage 1.2. Stage 1.1 forms an FCC with $N-4$ robots being in its interior (Fig. 3(b)). Stage 1.2 elects leaders A and B among the four corner robots of the FCC. The remaining two robots C, D on the FCC are also ranked first and second.

Stage 1.1. Let $\mathbf{P}$ be an SER of the robots in $\mathcal{Q}$ in any given $C_{non,init}$ on a grid (of unbounded size). All the robots of $\mathcal{Q}$ are either on the perimeter of $\mathbf{P}$ or in the interior of $\mathbf{P}$ (Fig. 4(b)). An SER, $\mathbf{P}$, is an FCC if and only if there are exactly four robots on the four corners of $\mathbf{P}$ and no other robot on the perimeter. The goal here is to form an FCC from the SER of $C_{non,init}$. A robot w is at a corner (or on a side) of the SER if all visible robots are within an axis-aligned angle of 90° (or 180° but not 90°). In the following discussion, w will take actions based on sides and corners of an SER. These refer to the smallest enclosing rectangle of the configuration visible from w; this rectangle may be the same as or a subset of the SER of the entire configuration.

The algorithm is as follows for each robot $w \in \mathcal{Q}$.

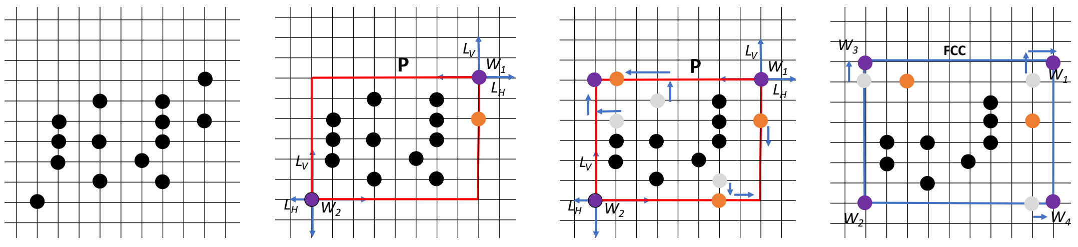


Figure 4: An illustration of Stage 1.1 forming an FCC: (a) $C_{non,init}$, (b and c) during Stage 1.1, and (d) after Stage 1.1. **P** is an SER (rectangular convex hull) of $C_{non,init}$.

- Let w be at a corner of the SER, then w changes its color to `rectangle-point`.
- Let w be a side robot on side S of the SER. If w sees a corner or side robot in one direction on S but nothing in the other direction, then it takes color `ready1` (yellow colored robot in Fig. 4(b)) and moves toward the empty corner. Suppose w sees no other robots on S and is closer to corner c_{near} of S than it is to corner c_{far} of S. Let I_S denote the set of interior robots closest to S. If I_S is empty or includes a robot whose projection to S is at equal or less distance to c_{far} than w, then w takes color `ready1` and moves toward c_{near}; otherwise, w takes color `ready1` and moves toward c_{far}. Now suppose w sees no other robots on S and is equidistant from both corners of S. If I_S has robots with projections to S in only one direction from w, then w takes color `ready1` and moves toward the corner in the opposite direction; otherwise, w takes color `ready1` and arbitrarily chooses one corner and moves toward it.
- Let w be an interior robot closest to side S of the SER and at an extreme toward a corner of S of the set I_S of robots closest to S, and let I_S contain no robots with color `ready1`. In this item, let corner robot mean a robot at a corner of the SER or a robot with color `rectangle-point` that is Manhattan distance 1 or 2 away from a corner of the SER. If S has one corner robot but no side robots and w is the robot in I_S closest to the empty corner, then w moves toward S. If S has one side robot x but no corner robots and x is closer to corner c_{near} of S than it is to corner c_{far} of S, then w does the following. If the projection of w on S is closer to c_{far} than x is and w is closer to c_{far} than other robots in I_S, then w moves toward S. (Robot x will move toward c_{near} and w will move toward c_{far}.) If the projection of w on S is closer to c_{near} than x is and w is closer to c_{far} than other robots in I_S, then w moves toward S. (Robot x will move toward c_{far} and w will move toward c_{near}.) In the above in this item, when the move of w toward S will place w on S, then w takes color `ready1` before the move, so w will have color `ready1` when it arrives on S. Fig. 4(c) illustrates these robots (yellow colored) moving to a side then toward a corner.
- Let w be a side robot not colored `ready1`, in a configuration with no interior robots, and closest to side S of the SER that has only one side robot and no corner robots or only one corner robot and no side robots, then w moves into the interior. This puts w into a position from which it can later move to a corner of S.
- If two robots were to head to the same rectangle point (one in horizontal direction and one in vertical direction) and they are at distance one from the rectangle point, then they use leader election to break symmetry (and avoid collision) so that one will continue moving towards the rectangle point. The leader election procedure described later in Stage 1.2 can be used for this symmetry breaking. The robot that loses the leader election changes its color back to `start`.
- If a robot w has color `rectangle-point` and sees a robot on one of its (axis-aligned) sides that does not have color `rectangle-point or ready1` (that is, a side robot), then w moves unit distance away from the side (toward the exterior) to be visible to the neighboring corner (see Fig. 4(d)).

- If a robot w has color `rectangle-point` and is not at a corner of the SER, then it identifies the robot with color `rectangle-point` at the neighboring corner and moves outward to align itself with that robot in a corner position. If w has two such non-aligned neighbors, then w picks one of them arbitrarily and moves to align with it.
- If a robot w has color `ready1` and is not on a side S of the SER, then w moves to place itself on S, keeping color `ready1`.

For any $C_{non,init}$ with $N = 3$, each robot will figure out that there are only three robots in $\mathcal{Q}$ while executing Stage 1.1. The robots then terminate forming a triangular configuration.

Lemma 3.1 *At the end of Stage 1.1: (i) for $N = 3$, the robots terminate during Stage 1.1, (ii) for $N \geq 4$, the FCC is correctly formed with $N - 4$ robots in its interior. The corner robots of the FCC have color* `rectangle-point` *while the interior robots have color* `ready1` *or* `start`. *Stage 1.1 finishes in $\mathcal{O}(N)$ epochs avoiding collisions, using randomization for symmetry breaking. The total number of colors used in Stage 1.1 is 3.*

Proof. For the SER of the robot configuration (starting from $C_{non,init}$), consider each corner c. If a robot is at c, then it will take color `rectangle-point` and will remain at a corner of the SER. (It may move, but will remain at a corner of the new configuration.) If no robot is at c, then consider one side S of that corner. If S has two or more side robots on it, then the one closest to c will take color `ready1` and move toward c. It will either reach c and take color `rectangle-point` or it will lose a randomized selection to a robot colored `ready1` on the adjacent side and that robot will reach c and take color `rectangle-point`. If c has no robot and S has only one side robot r, then if the corner at the other end of S has a robot, then r will take color `ready1` and move toward c as above, otherwise, an interior robot will move to S and the algorithm will proceed as above for two or more side robots. Each corner robot colored `rectangle-point` will move to see the two neighboring corners without being blocked by side robots. A robot will move at most twice after taking color `rectangle-point`. Collision avoidance is obvious since robots never move to the same grid point and towards each other (except for two robots moving to the same corner, and these use leader election described later in Stage 1.2 to avoid collision). For runtime, $\mathcal{O}(N)$ epochs are enough since even if the robots move sequentially, only at most eight robots compete to become rectangle points and four are successful. It is immediate that Stage 1.1 uses 3 colors `start`, `ready1`, and `rectangle-point`; the color `start` is the color of robots in C_{init}. □

Stage 1.2. After an FCC is formed, the goal is to elect two adjacent robots among the four on rectangle points of the FCC as first and second leaders, A and B, respectively. Leader A is picked, then leader B is picked, which is a neighbor of A on the SER. A is colored `leader1` and B is colored `leader2`. Two remaining rectangle points C, D are colored `defeated1` and `defeated2`, respectively, where C is the neighboring rectangle point of A in the FCC.

Leader Election. In the context of robots with lights, the *leader election* problem is as follows: Given a non-empty subset S of robots (indicated by their relative positions and light colors), select exactly one robot from S as the leader. On termination, the leader is colored `leader` and all other robots of S are colored `non-leader`.

Leader Election under Complete Visibility. Here we assume that all robots in the set S are visible to each other, so each competing robot is aware of $M = |S|$, the number of competing robots. The algorithm executes the following for each robot at each round.

- Toss a coin to select light color `leader` (resp., `non-leader`) with probability $\frac{1}{M}$ (resp., $1-\frac{1}{M}$).
- If there is exactly one robot in S with color `leader`, then terminate; otherwise repeat the previous step.

It is well-known that the above algorithm terminates with a leader in $O(\sigma \log M)$ rounds with probability $1 - \frac{1}{M^\sigma}$. Conversion of this algorithm to the $\mathcal{ASYNC}$ model is simply a matter of using lights (states) to synchronize robots into simulated rounds. In each simulated round (see Fig. 5), a

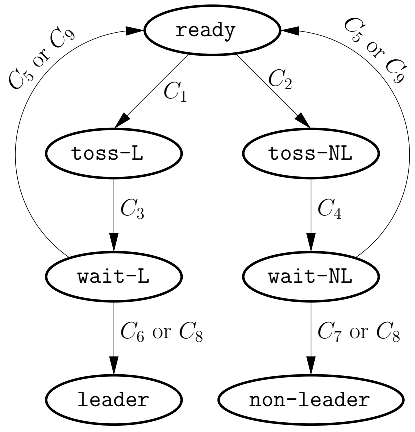


Figure 5: State diagram for $\mathcal{ASYNC}$ leader election when competing robots are visible to each other. The conditions C_1–C_9 are explained in the text describing the algorithm.

robot will cycle through a `ready` state, one of two "toss" states (`toss-L` and `toss-NL`) and one of two "wait" states (`wait-L` or `wait-NL`). From a "wait" state the robot either goes back to the `ready` state or terminates in one of two "terminal" states (indicated by lights `leader` and `non-leader`).

For the description below on leader election, we assume that robots start at the `ready` state and that every robot trying to be the leader can see every other robot trying to be the leader (complete visibility among the competitors). We discuss later how this leader election procedure is applied in the context of executing Stage 1.2 of our algorithm (where complete visibility among competitors is not assured).

When a robot is in the `ready` state and it sees all robots in either the `ready` or one of the "toss" states, it tosses a coin (as indicated in the algorithm above) and goes to state `toss-L` (resp., `toss-NL`) with probability $\frac{1}{M}$ (resp., $1 - \frac{1}{M}$); these transition conditions are denoted by C_1 and C_2, respectively in Fig. 5. No robot proceeds from a "toss" state until all robots are either in a "toss" state or a "wait" state. When a robot is in `toss-L` (resp., `toss-NL`) state and it sees all robots in either a "toss" state or a "wait" state, then it moves to state `wait-L` (resp., `wait-NL`); this is indicated by conditions C_3 and C_4 in Fig. 5. No robot proceeds from a "wait" state until all robots are in a "wait," "terminal" or `ready` state; here we use the term "terminal state" to denote either state "leader" or "non-leader." Specifically, if a robot r is in state `wait-L` (resp., `wait-NL`), and no robot is in a "toss" state, then it proceeds as follows. If there is a robot in state `ready`, then robot r too moves to state `ready` (Condition C_5 in Fig. 5). If there is a robot in a terminal state, then the robot r moves to `leader` (resp., `non-leader`) (conditions C_6 (resp., C_7) in Fig. 5). If no robot is in `ready` or a "terminal" state, then the robot r moves to state `leader` by condition C_8 (resp., `non-leader` by condition C_9) if it sees exactly one robot (including itself) in state `wait-L`; otherwise, robot r moves to state `ready` (condition C_9). The correctness of this algorithm follows from the fact that no new round starts until all robots have completed the current round. In fact, the time complexity of this $\mathcal{ASYNC}$ algorithm is the same as in the $\mathcal{FSYNC}$ model, with a round replaced by an epoch. The number of colors needed for the algorithm is 7, one for each state in Fig. 5.

Lemma 3.2 *Leader election among a set of M robots that are visible to each other can be performed on the $\mathcal{ASYNC}$ model with lights with 7 colors in $O(\sigma \log M)$ epochs, with probability $1 - \frac{1}{M^\sigma}$.*

For this paper we will require a special case of Lemma 3.2 with $M = 2$. For an overall problem size of N and "with high probability (whp)" indicating a probability of at least $1 - N^{-c}$ for constant $c > 1$, we use $\sigma = c \log N$ to obtain the following result.

Corollary 3.3 *Leader election among a set of 2 robots that are visible to each other can be performed on the $\mathcal{ASYNC}$ model with lights in $O(\log N)$ epochs, with probability $1 - \frac{1}{N^c}$.*

Leader Election on a Non-Empty Rectangle. Here $S = \{0, 1, 2, 3\}$ is a set of four robots such that for each $0 \leq i < 4$, robot i can see itself and robots $(i \pm 1) (\text{mod } 4)$. (In the following discussion we will write $i \pm 1$ to mean $(i \pm 1) (\text{mod } 4)$. Also the index i itself is only for our reference; the robots themselves are anonymous.) We will call the robots in set $\{i - 1, i, i + 1\}$ as the "neighbors" of i. As before, we consider the $\mathcal{FSYNC}$ case first. The basic algorithm is also the same as the one before (the standard slotted ALOHA-based as described in Wattenhofer [22]).

The robots will start with color `ready`, flip a coin in the first round and color themselves accordingly. We will call this new color as the "toss color." Specifically, the robots begin by coloring themselves `zero` (or `one`) with probability $\frac{1}{4}$ (or $\frac{3}{4}$). If there is exactly one robot with toss color `one`, then that will (subsequently) become the leader.

In the second round, each robot i will generate a composite color (x_i, y_i), where $x_i \in \{\texttt{zero}, \texttt{one}\}$ is its toss color. The "view color" $y_i \in \{0, 1, 2, 3\}$ of robot i is the number of robots among its neighbors $i - 1, i, i + 1$ that have "toss color" `one`.

At the third round, each robot can determine whether a leader has been found or another iteration of coin-flips is needed. The following lemma develops the conditions to make this determination.

Lemma 3.4 For $S = \{0, 1, 2, 3\}$ and any robot $i \in S$, the number of robots in S with toss color `one` is 0 (resp., 1, ≥ 2) iff the highest view color among the neighbors of i is 0 (resp., 1, ≥ 2).

Proof. We first prove that if the total number of robots with toss color `one` is $c \in \{0, 1, \geq 2\}$, then the largest view color that robot i sees is c.

- Case $c = 0$: Every robot must have a view color of 0.

- Case $c = 1$: If robot i is the robot with toss color `one`, then robots $i - 1, i, i + 1$ all have a view color 1. If robot $i - 1$ or $i + 1$ is the robot with toss color `one`, then that robot and robot i both have a view color of 1 while the other robot of the pair has a view color of 0. If robot $i + 2$ is the robot with toss color `one`, then robots $i - 1$ and $i + 1$ have view color 1 while robot i has view color 0. Across these instances, the largest view color that robot i sees is 1.

- Case $c \geq 2$: If at least two of the neighbors of robot i have toss color `one`, then i has a view color at least 2. If only one of the neighbors of robot i has toss color `one`, then $i + 2$ must have toss color `one`. Then if $i - 1$ or $i + 1$ is the neighbor with toss color `one`, then that robot will have view color 2; otherwise, i is the neighbor with toss color `one`, so $i - 1$ and $i + 1$ will have view color 2. Across these instances, the largest view color that robot i sees is at least 2.

We now prove the converse that if the largest view color that robot i sees is $c \in \{0, 1, \geq 2\}$, then the total number of robots with toss color `one` is c. Again we consider some cases.

- Case $c = 0$: If robot i sees a maximum view color of 0, then neighbor $j \in \{i - 1, i, i + 1\}$ has view color $y_j = 0$, so each robot $k \in \{j - 1, j, j + 1\}$ must have a toss color of `zero`. With addition and subtraction modulo 4, the set $\{j - 1, j, j + 1 \;:\; i - 1 \leq j \leq i + 1 \text{ and } 0 \leq i < 4\} = \{i - 2, i - 1, i, i + 1, i + 2 \;:\; 0 \leq i < 4\} = \{0, 1, 2, 3\}$ is the entire set of robots under consideration.

- Case $c = 1$: Suppose robot i sees a maximum view color of 1. If i has a toss color of `one`, then each of robots $i \pm 1$ must have a toss color of `zero` and so must robot $i + 2$; otherwise, the view color of $i \pm 1$ would be > 1. On the other hand, if i has a toss color of `zero`, then at most one of $i \pm 1$ has toss color `one`. If one does, then $i + 2$ has toss color `zero`; if neither does, then $i + 2$ has toss color `one` because i sees a view color 1. These two observations imply that if i that sees a maximum view color of 1, then exactly one among the four robots of S has toss color `one`.

- Case $c > 1$: If a robot i sees a view color $c \geq 2$, then at least two of its neighbors or its neighbors' neighbors must have a toss color of `one`. This set $\{j, j + 1, j - 1 \;:\; i - 1 \leq j \leq i + 1\} = \{i - 2, i - 1, i, i + 1, i + 2\} = \{0, 1, 2, 3\}$ is the entire set of robots under consideration.

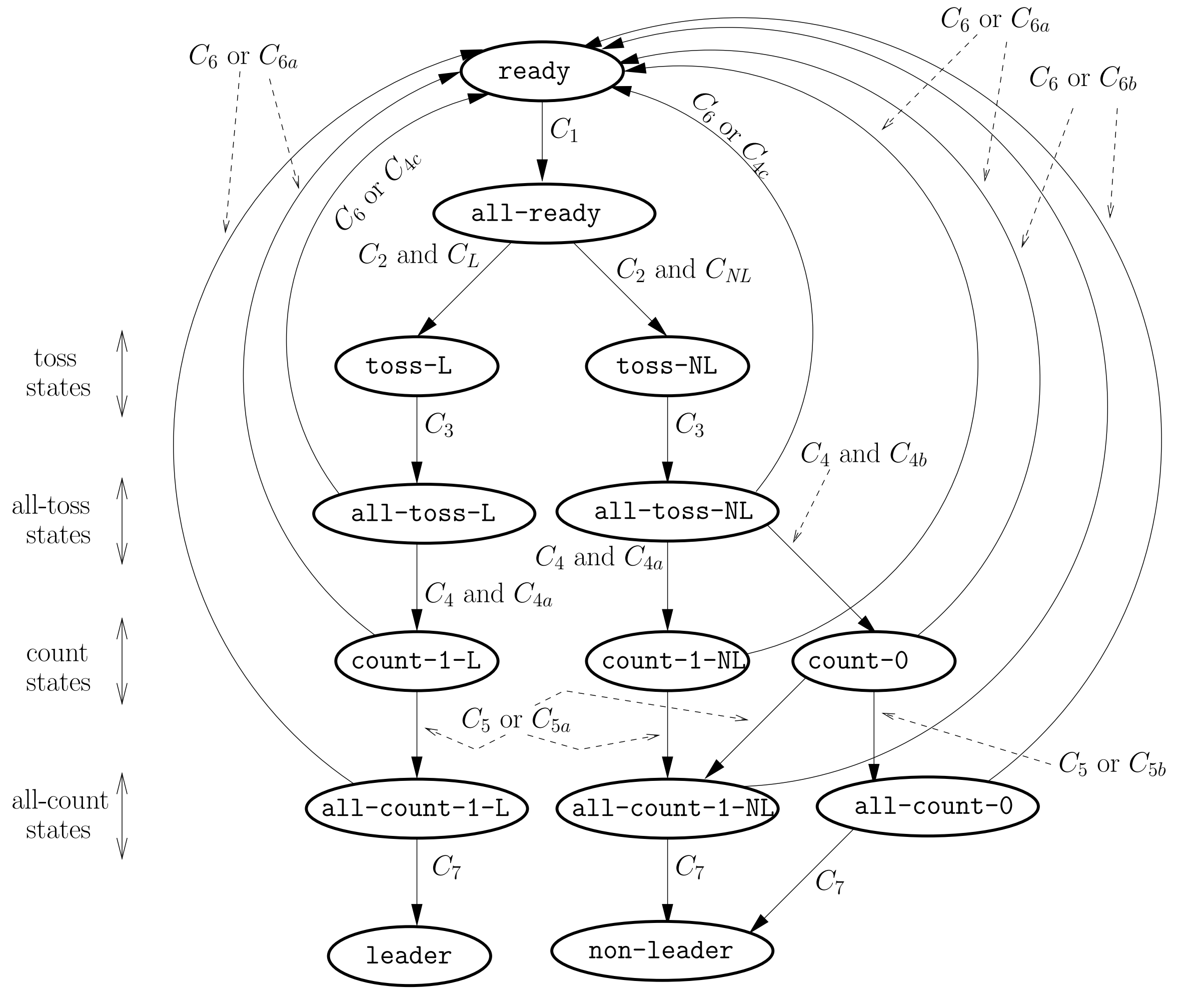


Figure 6: State diagram for $\mathcal{ASYNC}$ leader election when competing robots are placed at corners of a rectangle and two robots are visible to each other if they share a side in this rectangle (diagonal robots may not be able to see each other). The conditions used to label the edges are given in Table 2. Some of the states are collective names, for example states `toss-L` and `toss-NL` are collectively called "toss states;" these collective names are also shown in the figure. In addition we will also refer to states "1" and "0" states in the obvious way from the count and all-count states.

□

Lemma 3.4 establishes the criterion to terminate the leader election algorithm with an appropriate `leader` or `non-leader` flag or to iterate again. The extension of this algorithm to the $\mathcal{ASYNC}$ model follows the idea of Fig. 5.

Consider the end of Stage 1.1 in which all four corner robots of the FCC have color `rectangle-point` and all interior robots have color either `ready1` or `start`. Only the four corner robots of the FCC take part in the leader election process with color `rectangle-point`. Let A, C, D, B be four corners named consecutively in the clockwise direction starting from corner A. For A to start the leader election procedure, it has to see both of its neighbors B, C colored `rectangle-point`. This condition applies to B, C, D as well. Therefore, these four robots synchronize on the colors such that all of them start leader election with color `rectangle-point`. This is equivalent to having all four robots colored "`ready`" when starting leader election (Fig. 6). The robots can then transition to other colors as described in the leader election procedure. Each state in Fig. 6 corresponds to a light color.

Table 2: Conditions used in Fig. 6

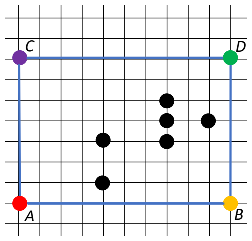
Symbol	Condition
C_1	Robot can see only `ready` or `all-ready` states
C_2	Robot can see only `all-ready` states
C_L	Robot has tossed a 1 (Leader)
C_{NL}	Robot has tossed a 0 (Non-Leader)
C_3	Robot can only see toss or all-toss states
C_4	Robot can only see all-toss states
C_{4a}	Robot can see exactly one robot (including itself) in state `all-toss-L`
C_{4b}	Robot cannot see any robots in state `all-toss-L`
C_{4c}	Robot can see more than one robot in `all-toss-L` state
C_5	Robot can only see count or all-count states
C_{5a}	Robot can see a robot (possibly itself) in a “1” state
C_{5b}	Robot cannot see any robot in a “1” state
C_6	Robot can see another robot in `ready` state
C_{6a}	Robot can see two or more robots in a “1” state
C_{6b}	Robot can cannot see any robots in “1” states
C_7	Robot can only see all-count states and the robots sees a “1” state

As noted earlier, robots start at state `ready`. When a robot i (say), that in state `ready`, sees itself and is two neighbors in state `ready` or `all-ready`, it moves to state `all-ready` and stays there until all its neighbors have moved to `all-ready`. At this point robot i has the assurance that all four robots competing in leader election are in states `ready` or `all-ready`. It now tosses a coin and moves to one of states `toss-L` or `toss-NL`. It is possible that at this point robot i is in a toss state, but its diagonally opposite robot $i+2$ (that may not be visible to i) is still in state `ready`. However, robots $i \pm 1$ cannot move to a toss state until robot $i+2$ moves to `all-ready`. Consequently, robot i cannot leave the toss state until robot $i+2$ moves to `all-ready`. Thus the addition of state `all-ready` serves to synchronize robots between “ready” and “toss” states (that is not automatic as diagonally opposite robots may not be visible to each other). Similarly, the all-toss states serves to assure a robot trying to move to a count state that all robots are in a toss state, and the all-count states assure robots that the toss of every robot has been factored in. At a count state a robot has a possibly partial count of the number of “leaders”; in an all-count state, each robot has an exact count of the number of “leaders.” If this count is greater than 1, then robots move to `ready` for the next iteration. When all robots are in an all-count state and see a “1” robot, then the leader has been found.

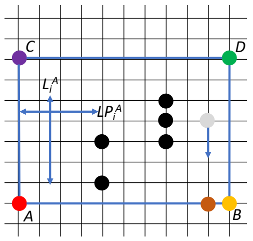
Thus, the leader election procedure above uses 14 colors to find a leader among the 4 corners A, B, C, D of the FCC. One color `ready` can overlap with `rectangle-point`, therefore, Stage 1.2 needs 13 additional colors (that are different from colors in Stage 1.1) to elect a leader among the 4 corners A, B, C, D of the FCC. However, the current configuration of robots may block diagonal robots of the FCC. To work as if all robots are visible for leader election, a robot must take cues from the two robots on its FCC sides (that are visible to it).

After one corner, say A (among $A, B, C,$ and D), is picked as a leader with color `leader1`, the procedure can be repeated to select other three robots among B, C, D with colors `leader2`, `defeated1`, and `defeated2`, respectively, as needed by our algorithm.

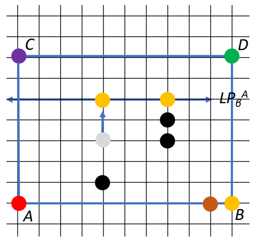
Consider the situation of one robot, say A, is picked as the first leader and colored `leader1`. The rest three robots B, C, D are colored `non-leader`. Let B, C be the neighbors of A in the FCC. We ask our algorithm to pick either B or C as the second leader and color `leader2`. We ask all three robots B, C, D to assume color `ready2`, transitioning from `non-leader`. This color signifies that they are now running the leader election algorithm to pick the second leader. Robot A colored `leader1` will not take part in the leader election process as it is already selected as a leader. B and C



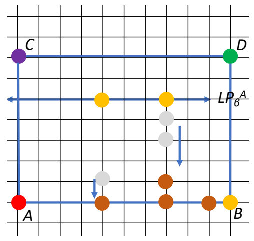
(a) After Stage 1



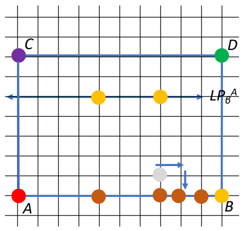
(b) After Stage 2.1



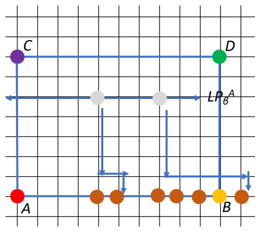
(c) After Stage 2.2



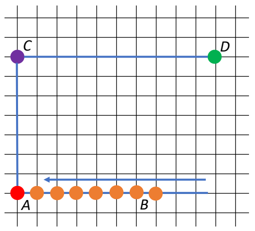
(d) After Stage 2.3



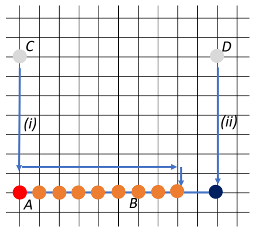
(e) After Stage 2.4



(f) After Stage 2.5



(g) After Stage 2.6



(h) After Stage 2.7

Figure 7: An illustration of the configuration after Stage 1 and the seven sub-stages of Stage 2: (a) after Stage 1, (b) after Stage 2.1, ..., (h) after Stage 2.7. The configuration shown for each sub-stage is achieved at the end of that sub-stage.

also know that A will not take part in the second leader election and communicate respectively with D regarding A's choice. Furthermore, since D is not the neighbor of A, it will help to select either B or C the second leader. Therefore, using one extra color `ready2`, the colors in the leader election process can be reused to elect the second leader. Let B be the second leader colored `leader2`. If D is the neighbor of B it assumes `defeated2`, otherwise it assumes `defeated1`, since it must be the neighbor of A. C assumes color `defeated1` if D is colored `defeated2` and vice-versa. Therefore, for C, D to pick colors, the leader election process does not need to be repeated.

Therefore, Stage 1.2 needs 3 colors to denote leaders B, C, D on top of 13 colors for selecting the first leader and one extra color while selecting the second leader. Therefore, in total Stage 1.2 needs $3+13+1 = 17$ colors that are different from 3 colors in Stage 1.1. The lemma below summarizes the results of Stage 1.2.

Lemma 3.5 *By the end of Stage 1.2, two robots A, B on adjacent rectangle points of an FCC are elected as the first (A) and the second (B) leaders and colored* `leader1` *and* `leader2`*, respectively, and the other two robots C, D are colored* `defeated1` *and* `defeated2`*, respectively. Stage 1.2 finishes in $\mathcal{O}(\log N)$ epochs avoiding collisions, using randomization for symmetry breaking. The total number of colors used in Stage 1.2 (that are different from the colors used in Stage 1.1) is 17.*

3.2 Stage 2: Line Formation for Non-collinear Configurations

At the end of Stage 1, A, B, C, D are the rectangle grid points (or robots) of an FCC colored `leader1`, `leader2`, `defeated1`, and `defeated2`, respectively. The point pairs (A, B), (B, D), (A, C) and (C, D) share sides of the FCC. All remaining $N - 4$ robots are in the interior of the FCC colored `start` or `ready` (see Fig. 3(b)). The goal in Stage 2 is to position all N robots on $\overrightarrow{AB}$ at distances $0, 1, 2, \ldots, N-2$, and $N'-1$ from A, where $N' \geq N$ is the smallest prime number greater or equal to N (Fig. 3(c)). Robot A does not move and robot B will be at distance either $N'-1$ or one of $1, 2, \ldots, N-2$. Note that N is not known to the robots.

A simple approach is to move robots sequentially. All robots on a parallel line closest to $\overrightarrow{AB}$ first move to $\overrightarrow{AB}$. After that the robots on the next parallel line move to $\overrightarrow{AB}$, and so on. This approach, however, needs $\mathcal{O}(N^2)$ time (note we consider $D \leq N$; with $D > N$, the bound becomes $\mathcal{O}(DN)$);

at most $N-2$ lines have at least a robot on each, and the robots on each line need to traverse distance $\mathcal{O}(N)$ to be positioned on $\overrightarrow{AB}$. This will make the overall runtime $\mathcal{O}(N^2)$. Therefore, the challenge is to devise an approach that finishes Stage 2 in $\mathcal{O}(N)$ epochs.

Our approach runs in seven sub-stages, Stages 2.1–2.7, each taking $\mathcal{O}(N)$ time, giving overall $\mathcal{O}(N)$ runtime for the algorithm. Fig. 7 illustrates the ideas behind each sub-stage. The main challenge we address here is how to make moves in parallel.

We need some notation. For a robot w, let L_H (L_V) denote the horizontal (vertical) line through it, as w's local coordinate system defines horizontal (vertical). Without loss of generality, the description below refers to L_H when describing a condition that could apply to either line, such as being parallel to $\overrightarrow{AB}$. Moreover, let L_0^A (L_0^B) be the line perpendicular to $\overrightarrow{AB}$ passing through A (B). Let the length of the line segment $\overline{AB}$ joining leaders A and B be $\mathsf{length}(\overline{AB}) = m$ units, i.e., there are $m-1$ grid points on $\overrightarrow{AB}$ between A and B. Let L_i^A be the line perpendicular to $\overrightarrow{AB}$ passing through the grid point on $\overrightarrow{AB}$ at distance i from A. Since $1 \leq i < m$, there will be $m-1$ perpendicular lines between A and B, with L_1^A closest to L_0^A and L_{m-1}^A closest to L_0^B. Let $\mathsf{length}(\overline{AC}) = n$ units. Let LP_i^A be a line parallel to $\overrightarrow{AB}$ at distance i from $\overrightarrow{AB}$ towards $\overrightarrow{CD}$. LP_1^A is closest to $\overrightarrow{AB}$ and LP_{n-1}^A is closest to $\overrightarrow{CD}$.

Stage 2.1. All the robots in the interior of the FCC are on the grid points of lines $L_1^A, \ldots, L_{m-1}^A$. Each line $L_i^A, 1 \leq i \leq m-1$, has no, one, or more robots positioned on its grid points. Let those be called no-robot, one-robot, or multi-robot lines, respectively, depending on the number of robots on them. Stage 2.1 moves the robots on one-robot lines to position them on $\overrightarrow{AB}$ (Fig. 7(b)) in $\mathcal{O}(N)$ epochs. The flow of Stage 2.1 is to sweep down each occupied row of the configuration until reaching $\overrightarrow{AB}$, moving each robot on a one-robot line down one row and coloring each robot on a multi-robot line as `multi-robot`. Due to the colors of A, B, C, D, the robots that move in this stage can always differentiate side $\overline{AB}$ from side $\overline{CD}$, and hence they can move to $\overline{AB}$. Let w be a robot with color `start` or `ready` that sees only colors `defeated1`, `defeated2`, or `multi-robot` on one side of L_H, where L_H is parallel to $\overrightarrow{AB}$. If w is the only robot on L_V and is not at a side of the visible SER, then it moves perpendicularly to L_H in the direction opposite of the side with colors `defeated1`, `defeated2`, or `multi-robot` (Lemma 2.1). If w is the only robot on L_V and is at a side of the visible SER, then it changes color to `on-AB`. If w is not the only robot on L_V, then it changes color to `multi-robot`. Stage 2.1 finishes when all robots on one-robot lines reach $\overrightarrow{AB}$ and assume color `on-AB`.

Lemma 3.6 *During Stage 2.1, the robots in the interior of the FCC that are on one-robot lines move to grid points on $\overrightarrow{AB}$ and take color* `on-AB`*. The remaining interior robots, on multi-robot lines, take color* `multi-robot` *and do not move. Stage 2.1 finishes in $\mathcal{O}(N)$ epochs avoiding collisions.*

Proof. Let LP_β^A be the line closest to $\overrightarrow{CD}$ that has (at least) a robot on it at the start of Stage 2.1. Let w be a robot on LP_β^A. The robots from $\overrightarrow{AB}$ to $LP_{\beta-1}^A$ are colored different than C, D, hence from Lemma 2.1, w can move to $LP_{\beta-1}^A$ if it is on a one-robot line or it can take color `multi-robot` if it is on a multi-robot line. Robot w can wait on $LP_{\beta-1}^A$ until there is no robot on LP_β^A or all on it are colored `multi-robot`. After this, the conditions on Lemma 2.1 are again satisfied for robots on $LP_{\beta-1}^A$, and they can move to $LP_{\beta-2}^A$ or take color `multi-robot`. This process then continues until all robots on one-robot lines reach $\overrightarrow{AB}$ and all robots on multi-robot lines have color `multi-robot`. Regarding runtime, since we assume $D = \mathcal{O}(N)$, any side of the FCC is $\mathcal{O}(N)$ long, and hence w needs to move for $\mathcal{O}(N)$ epochs to reach $\overrightarrow{AB}$. Since the moves of robots on one-robot lines are happening in parallel (with synchronization between two consecutive lines finishing in one epoch), Stage 2.1 finishes in $\mathcal{O}(N)$ epochs. Collisions are avoided since there is no other robot on the one-robot lines. ◻

Stage 2.2. At the beginning of Stage 2.2, all robots in the interior of the FCC are positioned on multi-robot lines and have color `multi-robot`. Let LP_β^A be the line closest to $\overrightarrow{CD}$ that has at least one robot colored `multi-robot` on it. For all the multi-robot lines that do not have a robot on LP^A_β, Stage 2.2 moves one robot each from those lines to position it on LP^A_β. At the end of Stage 2.2, the interior robots on LP^A_β have color `multi-robot-2`, while the other interior robots have color `multi-robot-1`. The flow of Stage 2.2 is to sweep up each occupied row of the FCC from $\overrightarrow{AB}$ to LP^A_β. Stage 2.2 is done as follows. Let w be a robot with color `multi-robot` that sees only colors `multi-robot-1`, `on-AB`, `leader1`, or `leader2` on one side of L_H. Call this side of L_H as the *bottom* side, and call the other side of L_H as the *top* side. Robot w checks the following conditions to determine whether to perform the corresponding action.

(a) If it sees a robot colored `multi-robot` on L_V in the top side, then w changes its color to `multi-robot-1` without moving.

(b) If it sees one or more robots colored `multi-robot` in the top side but no such robot on L_V in that side, then it moves perpendicularly to L_H into the top side, keeping its current color `multi-robot`. Robot w is now unit distance closer to LP^A_β.

(c) If it sees no robot in the top side except one each colored `defeated1` and `defeated2`, then w changes its color to `multi-robot-2` without moving. Robot w is in fact positioned on LP^A_β.

Case (c) makes sure that after w reaches LP^A_β, it stops moving and this way guaranteeing the end of Stage 2.2 for each robot on multi-robot lines. Fig. 7(c) illustrates what is achieved after Stage 2.2.

Lemma 3.7 *Let LP^A_β be the line parallel and closest to line $\overrightarrow{CD}$ of the FCC that has at least a robot colored* `multi-robot` *positioned on it at the end of Stage 2.1. At the end of Stage 2.2, a robot colored* `multi-robot-2` *from each multi-robot line is positioned on LP^A_β. The other robots take color* `multi-robot-1` *(without moving). Stage 2.2 finishes in $\mathcal{O}(N)$ epochs avoiding collisions.*

Proof. Stage 2.1 swept toward $\overrightarrow{AB}$, whereas Stage 2.2 sweeps toward $\overrightarrow{CD}$. At first, the robots closest to $\overrightarrow{AB}$ see the robots on one side colored differently than the robots on the other side. This satisfies the conditions of Lemma 2.1, so they either move keeping their color `multi-robot` or change their color to `multi-robot-1` without moving. This continues for each occupied row. After reaching LP^A_β, a robot sees only robots C, D on the top side and can change its color to `multi-robot-2` and stay on LP^A_β. Collisions are avoided since only one robot on each multi-robot line moves and the moving robot is the one that is closest to LP^A_β among the robots on that line. Regarding runtime, the robots on each line can move independently in parallel, and in $\mathcal{O}(N)$ epochs, they reach LP^A_β, since any side of the FCC is $D \leq \mathcal{O}(N)$. □

Stage 2.3. At the beginning of Stage 2.3, one robot from each multi-robot line is positioned on LP^A_β, the line closest to $\overrightarrow{CD}$ that has robot(s) in it, and has color `multi-robot-2`. Stage 2.3 moves all the robots on each multi-robot line L^A_k, except on LP^A_β, to position them on consecutive positions on L^A_k starting from $\overrightarrow{AB}$ (including the grid point on $\overline{AB}$). Fig. 7(d) illustrates what is achieved at the end of Stage 2.3. Note that in the beginning of Stage 2.3, the position on $\overline{AB}$ of each multi-robot line L^A_k is empty so a robot on L^A_k can be positioned on it.

Consider a multi-robot line L^A_k. Let L^A_j and L^A_l be two multi-robot lines closest to L^A_k, one on each side of L^A_k. Notice that all these lines are perpendicular to $\overrightarrow{AB}$. If L^A_k has no other multi-robot line on one or both sides of it, it can use $\overline{AC}$ (or $\overline{BD}$) as L^A_j (or L^A_l). Notice that the robots on L^A_k see all the robots on both L^A_j and L^A_l, in particular the robots with color `multi-robot-2`. They will use the robots colored `multi-robot-2` to orient themselves, determining the direction of $\overline{AB}$.

Stage 2.3 is done as follows. Let robot w on L^A_k have color `multi-robot-1`. If its neighbors on the line perpendicular to L^A_k have color `on-AB`, `leader1`, or `leader2`, then w changes its color to `on-AB`, as it has reached $\overline{AB}$. Otherwise, if the next grid point towards $\overline{AB}$ is empty, then w moves to that grid point, but if that grid point has a robot with color `multi-robot-consecutive` or `on-AB`, then w changes its color to `multi-robot-consecutive`.

Let robot w have color `multi-robot-2`. If all visible robots toward $\overline{AB}$ have colors from $\left\{\texttt{multi}-\texttt{robot}-\texttt{consecutive}, \texttt{on}-\texttt{AB}, \texttt{leader1}, \texttt{leader2}\right\}$, then w changes its color to `multi-robot-3`. This color change initiates Stage 2.4. Note that all these robots are still on LP_{β}^{A}.

Lemma 3.8 *At the end of Stage 2.3, all the robots on multi-robot lines, except those colored* `multi-robot-3`*, are positioned on those lines in consecutive positions starting from $\overrightarrow{AB}$ and colored* `multi-robot-consecutive` *or* `on-AB`*. The robots that started with color* `multi-robot-2` *have changed color to* `multi-robot-3` *without moving. Stage 2.3 finishes in $\mathcal{O}(N)$ epochs avoiding collisions.*

Proof. Consider the robots on L_k^A. Each robot on L_k^A sees the robots on L_j^A and L_l^A colored `multi-robot-2` (`multi-robot-2` colored robots are positioned on LP_{β}^{A}). Since the robots on LP_{β}^{A} do not move in this sub-stage, the robots on L_k^A see the robots on L_j^A and L_l^A colored `multi-robot-2` at all times. Now, each robot w on L_k^A satisfies the conditions of Lemma 2.2 and hence they can move on L_k^A toward $\overline{AB}$. This continues until w is next to a robot with color `multi-robot-consecutive` or `on-AB` when it stops and takes color `multi-robot-consecutive` or until w reaches $\overline{AB}$ when it stops and takes color `on-AB`. Collisions are avoided since each robots moves only if the neighboring grid point is empty. Runtime is $\mathcal{O}(N)$ rounds since there are at most $N-4$ robots on a multi-robot line and since $D = \mathcal{O}(N)$, they can relocate to consecutive positions in $\mathcal{O}(N)$ epochs (Lemma 2.3). The robots colored `multi-robot-2` can color `multi-robot-3` since after the robots on L_k^A moved to consecutive positions, a robot colored `multi-robot-2` on L_k^A sees the closest robot on L_k^A colored `multi-robot-consecutive`. □

Stage 2.4. At the beginning of Stage 2.4, all the robots in the interior of the FCC are positioned as follows for each multi-robot line: one robot is on LP_{β}^{A} colored `multi-robot-3`, one-robot is on $\overline{AB}$ colored `on-AB`, and the rest of the robots are on consecutive positions on those lines starting from $\overline{AB}$ colored `multi-robot-consecutive` (Fig. 7(d)).

Stage 2.4 moves all the robots colored `multi-robot-consecutive` to position them on $\overrightarrow{AB}$ colored `on-AB`. Note that some of these robots may be past B on $\overrightarrow{AB}$. The flow of this stage is for each multi-robot line to pipeline its robots down to LP_1^A then along this line in the direction from A to B until reaching an empty grid point on $\overrightarrow{AB}$. The robot will then drop onto $\overrightarrow{AB}$ and take color `on-AB`. The multi-robot lines do this in order starting from the line closest to B. Each robot with color `multi-robot-consecutive` can orient itself to recognize the direction toward $\overrightarrow{AB}$ by using robots colored `multi-robot-3`. Each robot on LP_1^A can recognize that fact because it can see robots with color `leader1` (A) and `leader2` (B).

Let w be a robot with color `multi-robot-consecutive`. If w is not on LP_1^A, then if the adjacent grid point toward $\overline{AB}$ is open, then w moves to that grid point. If w is on LP_1^A, then w checks if (i) the adjacent grid point on LP_1^A in the direction from A to B is open and (ii) no robot with color `multi-robot-consecutive` is present in the quadrant defined by the side of LP_1^A opposite to $\overline{AB}$ and the side of the line through it perpendicular to $\overline{AB}$ that is opposite from A. If these conditions are satisfied, then w changes its color to `moving-to-AB` and moves to that open grid point. This is the circumstance when all multi-robot lines toward B from w have already pipelined themselves onto LP_1^A and it is the turn of w's multi-robot line.

Let w be a robot with color `moving-to-AB`. If the adjacent grid point on $\overrightarrow{AB}$ is open, then change color to `on-AB` and move to that point. If the adjacent grid point on $\overrightarrow{AB}$ is not open and the adjacent grid point on LP_1^A in the direction from A to B is open, then w moves to that open point.

This process then continues for all multi-robot lines of Stage 2.3 with at least three robots on them (for a multi-robot-line with two robots, a robot reaches $\overrightarrow{AB}$ in Stage 2.3 and one is still on LP_{β}^{A} colored `multi-robot-3`). After reaching $\overrightarrow{AB}$, they assume color `on-AB`. Fig. 7(e) illustrates what Stage 2.4 achieves. Some of the robots that moved to $\overrightarrow{AB}$ during this substage may have moved beyond B, so the overall configuration may no longer be an FCC (though an SER still exists).

Lemma 3.9 *Stage 2.4 takes robots colored* `multi-robot-consecutive` *and positions them on* $\overrightarrow{AB}$ *with color* `on-AB`*. Stage 2.4 finishes in* $\mathcal{O}(N)$ *epochs avoiding collisions.*

Proof. The process starts from the multi-robot line closest to B and goes toward A. The robots on each line move to LP_1^A in a pipelined fashion (Lemma 2.3). After reaching LP_1^A, the robots see both A, B and can determine the direction to move, always in the direction from A to B. Collisions are avoided since robots move only if their next grid point is empty. Runtime of $\mathcal{O}(N)$ is also immediate. While some robot has color `multi-robot-consecutive`, one such robot changes color to `moving-to-AB` and starts moving along LP_1^A at least every other epoch. Also, with this pipelined movement along LP_1^A, each robot needs to move at most $\mathcal{O}(N)$ distance to find the first empty grid point on $\overrightarrow{AB}$ to jump from LP_1^A and take color `on-AB`. This is because $\mathsf{length}(\overline{AB}) = \mathcal{O}(N)$ and there are N robots. So within $\mathcal{O}(N)$ distance from A on $\overrightarrow{AB}$, all robots moving on Stage 2.4 are accommodated. $\square$

Stage 2.5. At the beginning of Stage 2.5, the only robots in the interior of the SER are on LP_β^A with color `multi-robot-3`. Each such robot w moves toward $\overline{AB}$ until it reaches LP_1^A then waits until all other robots of LP_β^A reach LP_1^A. When all are at LP_1^A, these robots see both C, D and the robots on $\overrightarrow{AB}$. Robot w then moves on LP_1^A in the direction from A toward B whenever the next grid point is empty in a pipelined fashion then changes color to `on-AB` and jumps to $\overrightarrow{AB}$ whenever the first empty grid point is available on $\overrightarrow{AB}$. Fig. 7(f) illustrates this sub-stage.

Lemma 3.10 *Stage 2.5 moves all the robots colored* `multi-robot-3` *on* LP_β^A *to* $\overrightarrow{AB}$ *and colors them* `on-AB`*. Stage 2.5 finishes in* $\mathcal{O}(N)$ *epochs avoiding collisions.*

Proof. The robots on LP_β^A can move to LP_1^A in $\mathcal{O}(N)$ epochs. After reaching LP_1^A, they see both A, B at all times and can move on LP_1^A in the direction from A towards B in a pipelined fashion until the first empty grid point on $\overrightarrow{AB}$ is found into which to jump and color `on-AB`. This whole process again takes $\mathcal{O}(N)$ epochs. Collisions are avoided by not moving to the next point until it is empty. $\square$

Stage 2.6. At the beginning of Stage 2.6, no robots are in the interior of the SER. The robots on $\overrightarrow{AB}$ may not be in consecutive positions starting from A. The goal in this stage is to move them on $\overrightarrow{AB}$ towards A so that they will occupy consecutive positions on $\overrightarrow{AB}$. Fig. 7(g) illustrates what this sub-stage achieves.

Let w be a robot on $\overrightarrow{AB}$ with color `on-AB` or `leader2`. It can see C and D and so determine the directions of A and B. If the adjacent grid point toward A on $\overrightarrow{AB}$ is empty, then w moves to that grid point. If the adjacent grid point toward A on $\overrightarrow{AB}$ is occupied by a robot with color `on-AB-consecutive` or `leader1`, then w changes color to `on-AB-consecutive`. The coloring process starts from A towards B and after the farthest robot on $\overrightarrow{AB}$ takes color `on-AB-consecutive`, Stage 2.6 finishes.

Lemma 3.11 *Stage 2.6 relocates the robots on* $\overrightarrow{AB}$ *to consecutive points of* $\overrightarrow{AB}$ *with color* `on-AB-consecutive`*. Stage 2.6 finishes in* $\mathcal{O}(N)$ *epochs avoiding collisions.*

Proof. The correctness is immediate. There are $N-2$ robots on $\overrightarrow{AB}$ at the beginning of Stage 2.6. Let B' be the robot that is farthest from A on $\overrightarrow{AB}$. B' knows that it is the farthest. The total length of $\overline{AB'}$ from A until B' is at most $\mathcal{O}(N)$. Therefore. moving the robots toward A takes $\mathcal{O}(N)$ epochs since robots move in a pipelined fashion (Lemma 2.3). Collisions are avoided similarly as in previous sub-stages by not moving until the next grid point is empty. $\square$

Stage 2.7. At the beginning of Stage 2.7, the robots on $\overrightarrow{AB}$ are in consecutive positions and only C, D are not on $\overrightarrow{AB}$. In this stage, C first moves to position itself on $\overrightarrow{AB}$ next to a robot colored `on-AB-consecutive`. D remains in its position until this is done. After that D can count the total number of robots, N, in the system. It computes the first prime number $N' \geq N$. It then moves first to line LP_1^A and then along line LP_1^A until the neighboring grid point on $\overline{AB}$ is at $N'-1$ distance away from A. It then jumps to $\overrightarrow{AB}$ and assumes color `endpoint1`. All N robots are now on $\overrightarrow{AB}$, with A colored `leader1`, the robot at distance $N'-1$ colored `endpoint1`, and the remaining $N-2$ have color `on-AB-consecutive`. All robots are in consecutive positions with the possible exception of the robot colored `endpoint1`. Fig. 7(h) illustrates this sub-stage. Stage 2.7 (and hence Stage 2) then finishes.

Lemma 3.12 *At the end of Stage 2, $N-1$ robots are on $N-1$ consecutive positions on $\overrightarrow{AB}$ starting from A, and one robot is on $\overrightarrow{AB}$ at distance $N'-1$ from A where $N' \geq N$ is the first prime number $\geq N$. Stage 2.7 finishes in $\mathcal{O}(N')$ epochs avoiding collisions. The total number of colors used in Stage 2 (that are different from the colors used in Stage 1) is 9.*

Proof. Similarly as in Lemma 3.11, C can move to $\overrightarrow{AB}$ in $\mathcal{O}(N)$ epochs. D can compute N as it sees all the robots on $\overrightarrow{AB}$. D can also compute what is its $N'-1$ position on $\overrightarrow{AB}$ until it is on $\overrightarrow{AB}$, which happens at the end when it jumps to the $(N'-1)$-th grid point from A towards B. Collision avoidance is immediate.

The 9 colors used throughout Stages 2.1 to 2.7 are as follows: `multi-robot`, `multi-robot-1`, `multi-robot-2`, `multi-robot-3`, `multi-robot-consecutive`, `on-AB`, `on-AB-consecutive`, `moving-to-AB`, and `endpoint1`. These colors are different from the colors used in Stage 1. □

3.3 Stages 1 and 2: Leader Election and Line Formation for Collinear Configurations

We discussed in Sections 3.1 and 3.2 Stages 1 and 2 for $C_{non,init}$. For $C_{col,init}$, Stages 1 and 2 can be combined together and simplified significantly. Note that $C_{col,init}$ only considers initial configurations where all robots are positioned on a single horizontal or vertical grid line. All other initial configurations, including all robots initially on a single line which is not a grid line, are handled in $C_{non,init}$. We first discuss how to handle collinear $C_{col,init}$ when $N \geq 4$. Let A, B be the endpoint robots on $C_{col,init}$. Let $r_1, \ldots, r_{N-2}$ be the $N-2$ robots between A and B on $\overline{AB}$, with r_1 closest to A and r_{N-2} closest to B. We have that $\mathsf{length}(\overline{AB}) = \mathcal{O}(N)$, since we are assuming that $D = \mathcal{O}(N)$. In $C_{col,init}$, all robots are colored `start`. Robots A, B pick color `ready3`. They do not move. The robots $r_1, \ldots, r_{N-2}$ do nothing. After A is colored `ready3`, r_1 picks color `collinear-moving` and moves to either of the two neighboring grid points not on $\overline{AB}$. Robot r_{N-2} also does the same after B is colored `ready3`. Let L_{up}, L_{down} be the two lines parallel to $\overline{AB}$ at unit distance each. Robots r_1, r_{N-2} reach to either of them, independently, after they move, so they may both be on the same line or one may be on L_{up} and the other on L_{down}. After that, each robot $r_2, \ldots, r_{N-3}$ picks color `collinear-moving` and moves to position itself on either of L_{up} or L_{down}.

After $r_1, \ldots, r_{N-2}$ move from $\overline{AB}$, A sees B (and vice-versa). Now A, B use the leader election procedure in Stage 1.2 to elect one as the first leader colored `leader1`. The other becomes the second leader colored `leader2`. Suppose A is the first leader colored `leader1` and B is the second leader colored `leader2`. A now elects two leaders C, D colored `defeated1` and `defeated2`, respectively, as follows. Let L_V be the line perpendicular to $\overline{AB}$ closest to A such that it has at least a robot on it. The single robot on L_V colors itself `defeated1`, and we call it C. After that, the robot next to C on line L_{up} or L_{down} where C belongs colors itself as `defeated2`, and we call it D. D is already further away from A than C; D does not need to move. If no other robots are on L_{up} or L_{down} with C, then the closest robot on the opposite line takes color `defeated2` and moves perpendicularly to $\overline{AB}$ to the same line as C; call this robot as D.

Suppose both C, D are on L_{up} after all this (C, D on L_{down} is analogous). The robots on L_{down} pick color `on-AB-1` and move to $\overline{AB}$. The robots on L_{up} except C, D also move to $\overline{AB}$ picking color `on-AB-1`. After this, all $N-2$ robots except C, D are on $\overline{AB}$. This provides a scenario similar to the one that is achieved after Stage 2.5 (Section 3.2). The robots then execute Stages 2.6 and 2.7 (except D moves first in Stage 2.7) to reach the configuration achieved after Stage 2.7 in Section

3.2. The resulting configuration is such that all N robots are on $\overrightarrow{AB}$ with A colored `leader1`, the robot at distance $N'-1$ from A is colored `endpoint1`, and the remaining $N-2$ robots are colored `on-AB-consecutive`.

We now discuss $C_{col,init}$ with $N \leq 3$. For $N = 3$, there is exactly one robot r_1 between A and B on $\overline{AB}$. A, B do not move. Therefore, when r_1 moves after it sees A or B colored `ready3`, a non-collinear triangle configuration is achieved. A, B see each other and only one robot on L_{up} and L_{down} combined and hence terminate. r_1 can also terminate since it sees only A, B. If $N = 2$, A sees the light of the only other robot B `ready3` and terminates. When $N = 1$, the only robot A can simply terminate since it sees no other robot.

Lemma 3.13 *For any $C_{col,init}$, the approach above achieves the following: (i) For $N = 1, 2$, robot(s) terminate without moving, (ii) for $N = 3$, the robots terminate in a triangular configuration; and (iii) for $N \geq 4$, the robots are positioned on a line $\overrightarrow{AB}$ satisfying Lemma 3.12. Runtime is $\mathcal{O}(N)$ epochs and no collisions. The total number of colors used is 23.*

Proof. The correctness proof is immediate from the description and the proofs of Stages 2.6 and 2.7. Regarding runtime, the process until selecting A, B, C, D as leaders finishes in $O(1)$ epochs. The robots on L_{up} and L_{down}, except C, D move back to $\overline{AB}$ again in $\mathcal{O}(1)$ epochs. After that moving the robots already on $\overline{AB}$ to consecutive positions and after that placing C, D on $\overline{AB}$ using the techniques of Stages 2.6 and 2.7, respectively, takes $\mathcal{O}(N)$ epochs. Thus, overall, for any $C_{col,init}$, this approach finishes in $\mathcal{O}(N)$ epochs. Collision avoidance is also immediate. The total number of colors used is 23: `start`, `ready3`, `collinear-moving`, `on-AB-1`, `endpoint1`, `on-AB-consecutive`, and the 17 colors of Stage 1.2 (the leader election starts with color `ready3` here, which we counted already) to elect the four leaders A, B, C, D with colors `leader1`, `leader2`, `defeated1`, and `defeated2`, respectively. □

3.4 Stage 3: Placement at Complete Visibility Points

Irrespective of whether the initial configuration is $C_{non,init}$ or $C_{col,init}$, the configuration at the end of Stage 2 has all N robots on $\overrightarrow{AB}$, with A colored `leader1`, the robot at distance $N'-1$ colored `endpoint1`, and the remaining $N-2$ robots colored `on-AB-consecutive`. Let $w_i \neq A$ denote the robot in this configuration on $\overrightarrow{AB}$ at distance i, where $2 \leq i \leq N-2$ or $i = N'-1$. The goal in Stage 3 is to reposition the robots that are on $\overrightarrow{AB}$ at the end of Stage 2 to guarantee a Complete Visibility configuration. Let $(0,0)$ denote the coordinates of A, so the initial coordinates of w_i are $(i, 0)$. We will establish that positioning each robot w_i at the target point of $(i, i^2 \mod N')$ achieves Complete Visibility.

Observe that the target points of A and $w_{N'-1}$ are $(0,0)$ and $(N'-1, 1)$, respectively. Robots will use A and $w_{N'-1}$ at their target points to orient themselves and determine N'. Let LP_1^A denote the line parallel to $\overrightarrow{AB}$ on one side to be determined. The flow of Stage 3 is to move all robots, other than A, first to LP_1^A then to sweep up, one row at a time, with each robot stopping at its target point.

Every (infinite) line on the grid divides it into two "half-grids." A given line and a grid point not on the line uniquely specify a particular half-grid. Coming back to the problem, recall that robot w_1 is adjacent to A. First w_1 changes color to `finalizing`, picks a grid point perpendicular to $\overrightarrow{AB}$, say γ, and moves to γ. This determines the half-grid to which the rest of the robots will move. Robot w_1 terminates, changing its color to `final`. We say that the coordinates of w_1 while terminating at grid point γ are $(1,1)$. Below, we will describe Stage 3 using the coordinate system determined by the positions of A and w_1. Other robots that can see landmarks in the configuration (such as A) will be able to discern this coordinate system. Let HP_γ denote the half-grid bordered by $\overrightarrow{AB}$ containing w_1. The other robots will move into half-grid HP_γ to reach on their final positions.

When w_i becomes active and recognizes w_1 with color `finalizing` or `final`, it takes color `finalizing` (except $w_{N'-1}$, which takes color `endpoint`) and moves to its neighboring grid point on LP_1^A in HP_γ. Robot w_i can make this decision because it will see robots only toward HP_γ, not in the half-grid on the other side of $\overrightarrow{AB}$. Robot w_i's coordinates now are $(i, 1)$. This way all robots

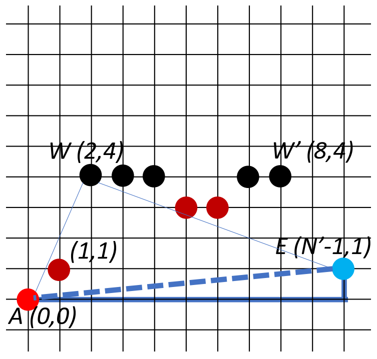


Figure 8: Placement to reach Complete Visibility points

that started on $\overrightarrow{AB}$ (except A) will eventually be positioned on LP_1^A between w_1 and $w_{N'-1}$ with color `finalizing`.

From here, Stage 3 sweeps up the rows parallel to $\overrightarrow{AB}$ as follows. Let w_i be a robot with color `finalizing`. Let L_H denote the axis-aligned line through w_i such that on one side of L_H (the *bottom side*) it sees no robots with color `finalizing` and sees at least one robot with color `leader1` (A), `endpoint` ($w_{N'-1}$), or `final`, and on the other side of L_H (the *top side*) it sees no robots with color `leader1` (A), `endpoint` ($w_{N'-1}$), or `final` but can see robots with color `finalizing`. If w_i sees robots with both colors `leader1` (A) and `endpoint` ($w_{N'-1}$) on the bottom side, then it can extract the coordinates of A and $w_{N'-1}$ and its own coordinates (i, y_i) in that system. If $y_i = i^2 \mod N'$, then w_i changes it color to `final` and terminates. Otherwise ($y_i \neq i^2 \mod N'$ or w_i does not see both A and $w_{N'-1}$), w_i moves to the adjacent grid point perpendicular to L_H in the top side. We will show that after all the robots assume color `final` and terminate, Complete Visibility is achieved.

Fig. 8 illustrates how this is done for a robot w at coordinate $(2, 4)$: Seeing both A and E (the robot colored `endpoint`), w can compute its coordinate $(2, 4)$, with respect to $(0, 0)$. Robot w' at coordinate $(8, 4)$ does not terminate at the grid point since (i) if it sees both A and E, then the coordinate computed, $(i', y'_i) = (8, 8^2 \mod 11) = (8, 9)$, does not match $(8, 4)$, (ii) if it does not see both A and E, then its current position is not the Complete Visibility position.

Correctness and Runtime of Stage 3. We first prove the correctness of Stage 3 meaning that a Complete Visibility configuration is indeed achieved. We will then prove the runtime and area bounds. For the correctness proof, we need the following theorem.

Theorem 3.14 (Roth [14]) *Suppose M is a prime number. Consider a 2-dimensional $M \times M$ grid G of M^2 grid points with the bottom-left grid point denoted as $(0, 0)$ and the top-right grid point denoted as $(M-1, M-1)$. The set of M grid points $(i, i^2 \mod M)$, for $0 \leq i < M$, on G contains no three collinear grid points.*

What essentially Theorem 3.14 means is that, for any prime number M, if the robots can be put on the M grid points $(i, i^2 \mod M), 0 \leq i < M$, then Complete Visibility is solved. However, there are several challenges in our model: robots do not know N and robots do not have a compass or agreement on the global coordinate system. We show that despite these limitations, robots can move to the grid points satisfying Theorem 3.14, correctly solving Complete Visibility after Stage 3. The following lemma helps in the correctness proof.

Lemma 3.15 *Consider an $M \times M$ grid G of M^2 grid points as in Theorem 3.14 with the bottom-left grid point denoted as $(0, 0)$ and the top-right grid point denoted as $(M-1, M-1)$. Let LP_j^A be the line connecting grid points $(0, j)$ and $(M-1, j)$, for $0 \leq j \leq M-1$, with LP_0^A being $\overleftrightarrow{AB}$. Suppose, for $0 \leq k \leq M-1$, where $0 \leq k^2 \mod M \leq j-1$, the robots on $LP_0^A, \ldots, LP_{j-1}^A$ are on the grid points $(k, k^2 \mod M)$ of G and the rest of the robots are on LP_j^A. Any robot w on LP_j^A can determine in Stage 3 whether it is on a grid point $(k, k^2 \mod M)$ of G.*

Proof. Since all the robots on $LP_0^A = (\overrightarrow{AB}), \dots, LP_{j-1}^A$ are on the grid points $(k, k^2 \mod M), 0 \leq k \leq M-1$, and no robot is on any other grid point of $LP_0^A, \dots, LP_{j-1}^A$, from Theorem 3.14, they must see each other (i.e., no three are collinear among robots already placed on their final positions). If a robot w on LP_j^A is on grid point $(k, k^2 \mod M)$ such that $j = k^2 \mod M$, again from Theorem 3.14, it must see all robots on $LP_0^A, \dots, LP_{j-1}^A$. We have that in Theorem 3.14, one robot is on LP_0^A at $(0,0)$ (which is robot A colored `leader1`) and two robots are on LP_1^A, one at $(1,1)$ and one at $(M-1,1)$. The robot at $(M-1,1)$ is colored `endpoint`. Let that robot be denoted as E. When w sees both A and E, it can compute M and the coordinates of A and E and its offset k from A. With k and M, w can compute whether $(k, k^2 \bmod M)$ is its current grid point. □

We also need the following theorem for the runtime proof.

Theorem 3.16 (Bertrand's postulate) *For any integer $N > 3$, there exists at least one prime number N' with $N < N' < 2N-2$. For any integer $N > 1$, $N < N' < 2N$.*

Lemma 3.17 *At the end of Stage 3, all robots are positioned on grid points solving* Complete Visibility. *Stage 3 finishes in $\mathcal{O}(N)$ epochs avoiding collisions. The area occupied by the robots in the final configuration is $\mathcal{O}(N^2)$. The total number of colors used in Stage 3 (that are different from the colors used in Stages 1 and 2) is 3.*

Proof. Note that in the beginning of Stage 3, all robots are positioned on an axis-aligned line $\overrightarrow{AB}$, starting from A in the direction from A to B. Let the position of A be $(0,0)$. At least $N-2$ other robots are on consecutive grid points. Let those points be $(1,0), \dots, (N-2,0)$. For N' the first prime number greater than or equal to N, $(N'-1,0)$ also has a robot on it. Consider A's position to be the origin of the global coordinate system with $\overleftrightarrow{AB}$ as one axis.

Let HP_γ be one half-grid bordered by $\overleftrightarrow{AB}$. If each robot on $(i,0)$, $0 \leq i < N'$, can move toward HP_γ perpendicularly to $\overrightarrow{AB}$ and position itself on the grid point at distance $i^2 \mod N'$, Theorem 3.14 is satisfied and Complete Visibility is solved.

The move of robot w_1 to $(1,1)$ provides a direction for robots $w_2, \dots$ to move, so that they can all move to the same half-grid HP_γ. Since $1 \mod N' = 1$ for any $N' > 1$, terminating w_1 at $(1,1)$ satisfies $(i, i^2 \mod N')$. All robots, except A, then move to LP_1^A (which has w_1 on it) and no robot moves to LP_2^A until all robots of $\overrightarrow{AB}$ reach LP_1^A. A remains on $\overrightarrow{AB}$, providing $(0,0)$. Robot $w_{N'-1}$ terminating on LP_1^A satisfies $(N'-1, (N'-1)^2 \mod N') = (N'-1, 1)$, since for any number $M' > 1$, $(M'-1)^2 \mod M' = 1$. After all robots move to LP_2^A, Lemma 3.15 is satisfied for each robot. If a robot w is at a coordinate $(i, i^2 \mod N')$, it terminates assuming color `final`. Otherwise, it continues moving to LP_3^A and beyond until it can terminate satisfying Lemma 3.15. This way all robots settle at the grid points of Theorem 3.14, achieving Complete Visibility. Collision avoidance is immediate since each robot moves on a line alone.

Regarding runtime, in one epoch, w_1 moves to LP_1^A. In the second epoch, w_1 terminates. In the same epoch, all the robots on $\overrightarrow{AB}$, except A, move to LP_1^A. In the third epoch, $w_1, w_{N'-1}$ terminate on LP_1^A and the rest move to LP_2^A. Since there are $N \leq N'$ robots in $\mathcal{Q}$ and $(i^2 \mod N') \leq N'-1$, the robots reach at most $LP_{N'-1}^A$. Therefore, robots reach $LP_{N'-1}^A$ at epoch N'. The robots that reach $LP_{N'-1}^A$ terminate at epoch $N'+1$. Therefore, Stage 3 finishes in $N'+1$ epochs. Using Theorem 3.16, $N \leq N' < 2N-2$. Therefore, Stage 3 finishes in $\mathcal{O}(N)$ epochs.

Regarding area, in the beginning of Stage 3, all the robots are on $\overrightarrow{AB}$ within N' distance. During Stage 3, robots move perpendicular to $\overrightarrow{AB}$ on a half-grid. A robot may move perpendicularly at most $(i^2 \mod N') \leq N'-1$. Therefore, the area of the final configuration is bounded by $N' \times (N'-1) \leq 2N \times (2N-1) \leq 4N^2 = \mathcal{O}(N^2)$, since $N' \leq 2N$.

Regarding the number of colors, Stage 3 uses 3 colors that are different from the colors used in both Stages 1 and 2, which are `finalizing`, `final`, and `endpoint`. □

3.5 Overall Correctness of the Algorithm

We showed so far that if each stage (and sub-stage) achieves the configurations as required to start the next stage (and sub-stage), COMPLETE VISIBILITY is guaranteed in our algorithm. That is indeed the case: The robots can correctly distinguish each stage (and sub-stage) during the execution and they work as intended.

Lemma 3.18 *Stages 1–3 execute sequentially one after another in the algorithm.*

Proof. Consider first any $C_{non,init}$. Stage 1 forms an FCC with four robots A, B, C, D on it. A is colored `leader1`, B `leader2`, C `defeated1`, and D `defeated2`. B (C) is adjacent to A (D). In Stage 2, the robots in the interior of the FCC that are the first to move (or change color without moving) must see both C, D colored `defeated1` and `defeated2`, respectively. This happens only after Stage 1.2 (and hence Stage 1) is finished with D colored `defeated2`. For $C_{col,init}$, when A, B, C, D pick color, all others are colored `collinear-moving` (different than they would have been colored at end of Stage 1 for $C_{non,init}$) and hence they can continue until the desired line configuration is achieved. For any C_{init} (i.e., collinear and non-collinear), Stage 3 is started only after all robots are in a collinear configuration with specific colors. In Stage 3, w_1, adjacent to A, moves first. For w_1 to move, it should not see C or D. D is the last to move to $\overrightarrow{AB}$ in Stage 2 and as soon as D moves to $\overline{AB}$, Stage 2 is finished. Therefore, Stage 3 executes only after Stage 2 finishes. □

Lemma 3.19 *Stages 1.1, 1.2, 2.1–2.7 correctly finish one after another in the algorithm.*

Proof. Stage 1.1 finishes after four robots, $r_1, \ldots, r_4$, are on an FCC and colored `rectangle-point`. Stage 1.2 cannot start until at least three rectangle points of the FCC have color `rectangle-point`. If Stage 1.2 starts after all $r_1, \ldots, r_4$ colored `rectangle-point`, we are done. Otherwise, Stage 1.2 must be started by r_i that sees both of its neighboring rectangle points colored `rectangle-point`. In this case, r_{i-1} and r_{i+1} detect that one of their neighbors is still not colored `rectangle-point` and do not take part in the Stage 1.2 procedure started by r_i. Lemma 3.18 establishes the sequentiality between Stages 1.2 and 2.1. Stage 2.2 starts only after all robots in the interior of the FCC are colored `multi-robot`, since the robots closest to $\overrightarrow{AB}$ start this process. For Stage 2.3, the robots on LP^A_β needs to be colored `multi-robot-2`. If robots on any line L^A_i start Stage 2.3 before Stage 2.2 finishes, the robots on L^A_i later wait for others to finish Stage 2.3, since they are independent of each other. Similarly, Stage 2.4 is synchronized from L^A_i close to B towards A and hence it is synchronized. For Stage 2.5, the only robots in the interior of the FCC should be on LP^A_β with color `multi-robot-3`. For Stage 2.6, there should be no robot in the interior of the SER, i.e., all robots on $\overrightarrow{AB}$ except C and D. For Stage 3, C and D should also be on $\overrightarrow{AB}$. This happens after D moves to $\overrightarrow{AB}$ in Stage 2.7 (Lemma 3.18). □

Proof of Theorem 1.1. For any $\mathbf{C}_{init}$ with diameter $D = \mathcal{O}(N)$, we have Theorem 1.1 with runtime $\mathcal{O}(N)$ epochs combining the results of Lemmas 3.1–3.13 and 3.17–3.19, **since** $N' \leq 2N$ **from Theorem 3.16**. For any C_{init} with $D = \Omega(N)$, running the techniques for Stages 1–3 only increases the runtime from $\mathcal{O}(N)$ epochs to $\mathcal{O}(D)$ epochs. In other words, $D = \Omega(N)$ does not have impact on the correctness of the algorithm. The area bound is immediate from Lemma 3.17 and is independent of D.

The total number of colors used throughout the algorithm depends on whether leader election is needed or not. Furthermore, irrespective of leader election, the number of colors used in our algorithm is the maximum between the number of colors used starting from non-collinear initial configurations $C_{non,init}$ and starting from collinear initial configurations $C_{col,init}$. The main reason is that Stages 1 and 2 are different for $C_{non,init}$ and $C_{col,init}$ and Stage 3 is common. We first describe the number of colors needed with leader election and then without leader election.

With leader election, for $C_{non,init}$, we need 32 colors as follows.

- 20 colors in Stage 1, which is the combination of

  - 3 colors in Stage 1.1 (Lemma 3.1).
  - 17 colors in Stage 1.2 (Lemma 3.5), that are different from the colors used in Stage 1.1.

- 9 colors in Stage 2 (Lemma 3.12), that are different from the colors in Stage 1.
- 3 colors in Stage 3 (Lemma 3.17), that are different from the colors in Stages 1 and 2.

With leader election, for $C_{col,init}$, we need 26 colors as follows.

- 23 colors in the combined Stages 1 and 2 (Lemma 3.13).
- 3 colors in Stage 3 (Lemma 3.17), that are different from the colors in the combined Stages 1 and 2.

Thus, taking the maximum on the colors used for $C_{non,init}$ and $C_{col,init}$, our algorithm needs 32 colors, when leader election is required.

We now discuss the number of colors required if no leader election is required. Without leader election, for $C_{non,init}$, we need 17 colors as follows.

- 5 colors in Stage 1, which is the combination of 4 colors to differentiate A, B, C, D (4 leaders) and 1 color `start` in C_{init}.
- 9 colors in Stage 2 (Lemma 3.12), that are different from the colors in Stage 1.
- 3 colors in Stage 3 (Lemma 3.17), that are different from the colors in Stages 1 and 2.

Without leader election, for $C_{col,init}$, we need 12 colors as follows.

- 9 colors in the combined Stages 1 and 2, since 14 colors to pick the first and second leaders are not required in the combined 23 colors with leader election.
- 3 colors in Stage 3 (Lemma 3.17), that are different from the colors in the combined Stages 1 and 2.

Therefore, taking the maximum on the colors used for $C_{non,init}$ and $C_{col,init}$, our algorithm needs 17 colors, when leader election is not required.

Overall correctness is then given by Lemmas 3.18 and 3.19. Theorem 1.1 follows combining the runtime, number of colors, and correctness results. □

4 Concluding Remarks

We have presented an $\mathcal{O}(\max\{D, N\})$-time, $\mathcal{O}(N^2)$-area algorithm for Complete Visibility for $N \geq 1$ robots with lights in the $\mathcal{ASYNC}$ setting on a grid (of unbounded size), a natural discretization of the 2-dimensional plane, where D is the diameter of the initial configuration. The number of colors in our algorithm is 17 if leader election is not needed and 32 otherwise. We also proved a time lower bound of $\Omega(N)$ and an area lower bound of $\Omega(N^2)$, irrespective of whether unlimited number of colors are available and leader election is not required, which shows that our algorithm is optimal for $D = \mathcal{O}(N)$. The best previously known 11-color deterministic algorithm has $\Omega(\max\{DN, N^2\})$ runtime and $\Omega(N^2)$ area.

Several questions remain open. We focused only on minimizing time and area but not the number of colors used. A better analysis would provide smaller number of colors than the current 32 when leader election is needed and 17 when leader election is not needed. Therefore, one interesting open question is can the number of colors be minimized to 2 (optimal) or close to 2 for Complete Visibility on a grid (of unbounded size). Another direction is to see whether robot faults (crash and/or byzantine) can be handled. The proposed algorithm and the previous algorithm do not consider faults.

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